Zero-Divisor Graphs of Direct Product Rings

Emily Salzman

Carthage College

esalzman@carthage.edu

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Abstract

This research investigates zero-divisor graphs of direct product rings. In zero-divisor graphs, if elements are connected with an edge, they will multiply to the additive identity element of direct product rings, $(0, 0)$. We look at graphs of the form $\boldsymbol{Z}_{2p} \times \boldsymbol{Z}_{2q}$, where p and q are prime numbers greater than 2. In order to generalize the structure of these graphs, many specific examples of $\boldsymbol{Z}_{2p} \times \boldsymbol{Z}_{2q}$ graphs are analyzed to better understand the common form. Upon investigating these rings, many "clumps", or "families", of zero-divisors appeared. Certain families will always connect to other families, and some families will never connect. This research investigates which families connect, making it possible to generalize the structure of these graphs.

1 Introduction

The following research investigates zero-divisor graphs in rings under modular multiplication. In the real number system, if $a * b = 0$, then either a or b must be 0. However, under modular multiplication, it is possible to multiply two non-zero numbers and have the product equal zero. When this happens, those numbers multiplied together are considered zero-divisors. Once we define all of the zero-divisors in these rings, we can visually represent them through the use of graphs. In zero-divisor graphs, the vertices are the zero-divisors, and the edges connect the zero-divisor pairs. After many specific examples, we were able to generalize the form of zero-divisor graphs of certain rings. The rings we examined were the direct product rings $\boldsymbol{Z}_{2p} \times \boldsymbol{Z}_{2q}$, where p and q are prime numbers. Within each direct product ring, there are certain families of zero-divisors that will always connect to other families of zero-divisors, making it possible to generalize the form of the zero-divisor graphs.

2 Definitions and Development

The following definitions and examples are crucial to understanding this research.

Definition 1 Let $a, b,$ and n be integers where $n > 0$. We say that a is congruent to $b \text{ modulo } n$, provided n divides $b - a$. This is called **modular arithmetic.**

Definition 2 Given two sets A and B, we may form from these the set of ordered pairs (a, b), $a \in A,\ b \in B$. These ordered pairs will be the elements of a new ring, the **direct product** $A \times B$, if we define our product by the rule $(a_1, b_1)(a_2, b_2) = (a_1a_2, b_1b_2)$.

It is important to take note of the fact that A and B may have different operations. In this research, the operation we are working with is modular multiplication, and rings A and B will have the same operation, but the modulus will differ.

Definition 3 Let x be an element of $\boldsymbol{Z}_n$. If there exists an element y in $\boldsymbol{Z}_n$ where $xy = 0$, then x is a **zero-divisor.**

Since we are looking at direct product rings, zero-divisors are defined as elements which are congruent to the ordered pair $(0, 0)$ when multiplied together. The element $(0, 0)$ itself is not considered a zero-divisor.

Definition 4 A **graph** consists of points (vertices), and connections (edges), which are indicated by line segments joining certain pairs of vertices.

In this research, we are exploring zero-divisor graphs, which are graphs where the vertices are the zero-divisors. The point of this research is to discover a general structure of these graphs. Let us look at an example of the zero-divisor graph of $\boldsymbol{Z}_2 \times \boldsymbol{Z}_3$.

Example 1 The zero-divisors of $\boldsymbol{Z}_2 \times \boldsymbol{Z}_3$ are the elements $(0, 1), (1, 0)$, and $(0, 2)$. These will be the vertices of the zero-divisor graph. When multiplied together, $(0, 1) * (1, 0) = (0, 0)$, and $(0, 2) * (1, 0) = (0, 0)$. However, $(0, 2) * (0, 1) \neq (0, 0)$, so $(0, 2)$ and $(0, 1)$ will not be connected with an edge. See Figure 1.

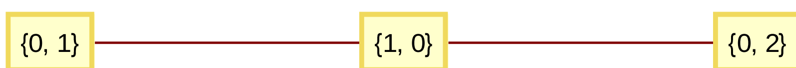

Figure 1: Zero-Divisor Graph of $\boldsymbol{Z}_2 \times \boldsymbol{Z}_3$

Example 2 A slightly larger ring, $\boldsymbol{Z}_5 \times \boldsymbol{Z}_6$, has more zero-divisors, 21 to be exact. There are many more vertices and edges in this graph, as can be seen in Figure 2. These examples are for background information only; the remainder of this research deals only with rings of the form $\boldsymbol{Z}_{2p} \times \boldsymbol{Z}_{2q}$ where p and q are prime numbers.

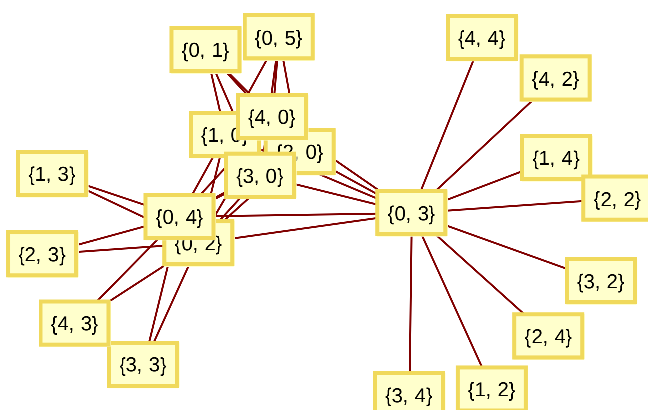

Figure 2: Zero-Divisor Graph of $\boldsymbol{Z}_5 \times \boldsymbol{Z}_6$

3 Results

We first investigated specific $\boldsymbol{Z}_{2p} \times \boldsymbol{Z}_{2q}$ rings, starting with smaller rings and building up to larger rings to try to find similarities. Some similarities were very evident. For example, it is clear that any of the elements $(x, 0)$ and $(0, y)$ will be zero divisors in any direct product ring. This is due to the fact that $(x, 0) * (0, y) = (0, 0)$. However, the elements $(x, 0)$ and $(0, y)$ may connect to other zero-divisors as well. We will investigate many elements and determine which zero-divisors connect by looking at specific examples of $\boldsymbol{Z}_{2p} \times \boldsymbol{Z}_{2q}$ rings.

Let us start with a fairly simple example: $\boldsymbol{Z}_6 \times \boldsymbol{Z}_{10}$. As you can see in Figure 3, there is a symmetrical look to this graph. We can see some “clumps” (henceforth referred to as “families”) of numbers forming, but it may be unclear why or how they are related.

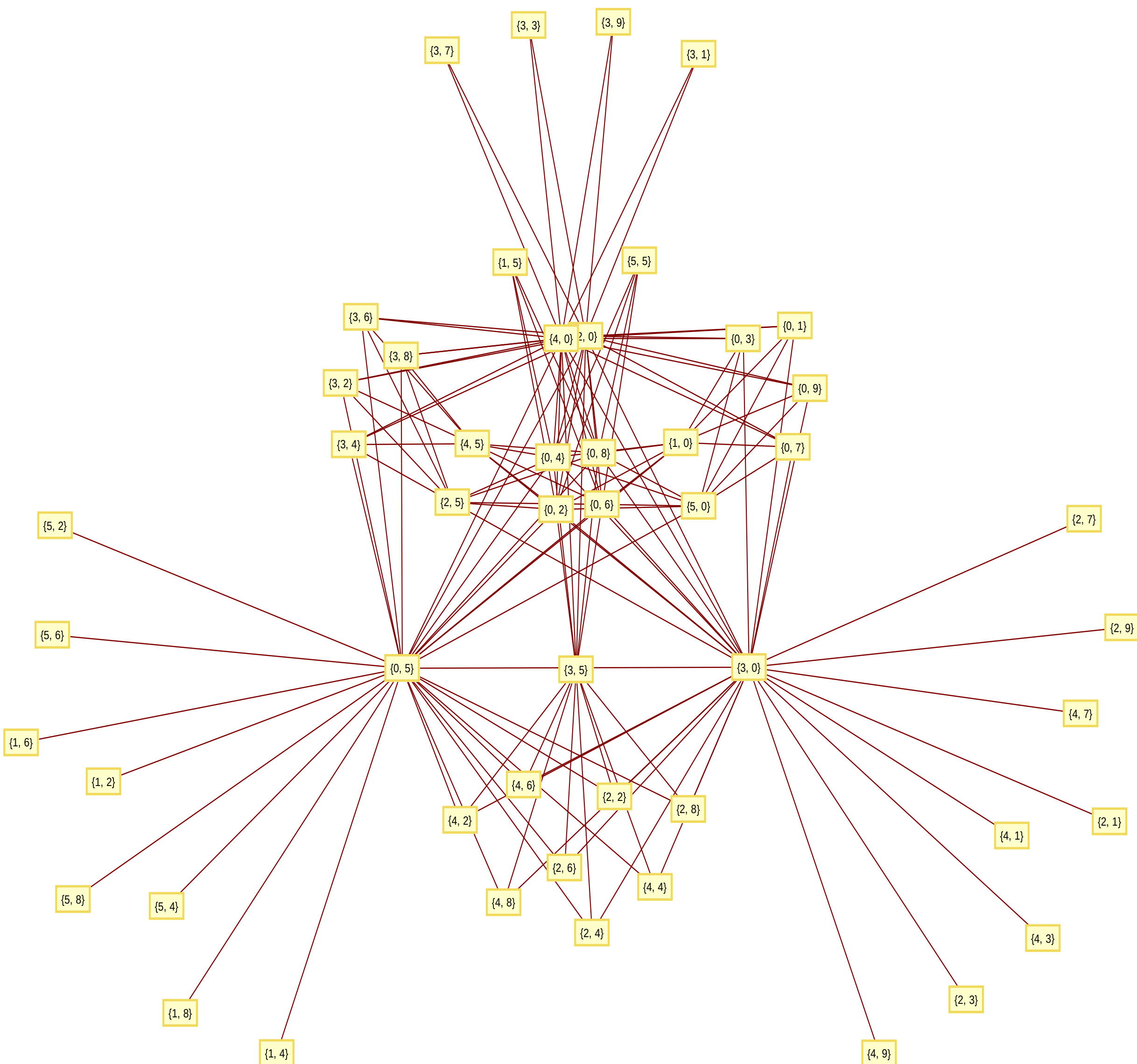

Figure 3: Zero-Divisor Graph of $\boldsymbol{Z}_6 \times \boldsymbol{Z}_{10}$

To get a better idea of the forming families, let us look at a slightly larger ring: $\boldsymbol{Z}_{10} \times \boldsymbol{Z}_{14}$, as shown in Figure 4. It is clear here that the center elements of $(0, 7), (5, 7)$, and $(5, 0)$ are the "core" elements; they have the most edges and each make up their own family. Some other families become apparent in Figure 4, but it might still be too difficult to count the families or make much sense out of them.

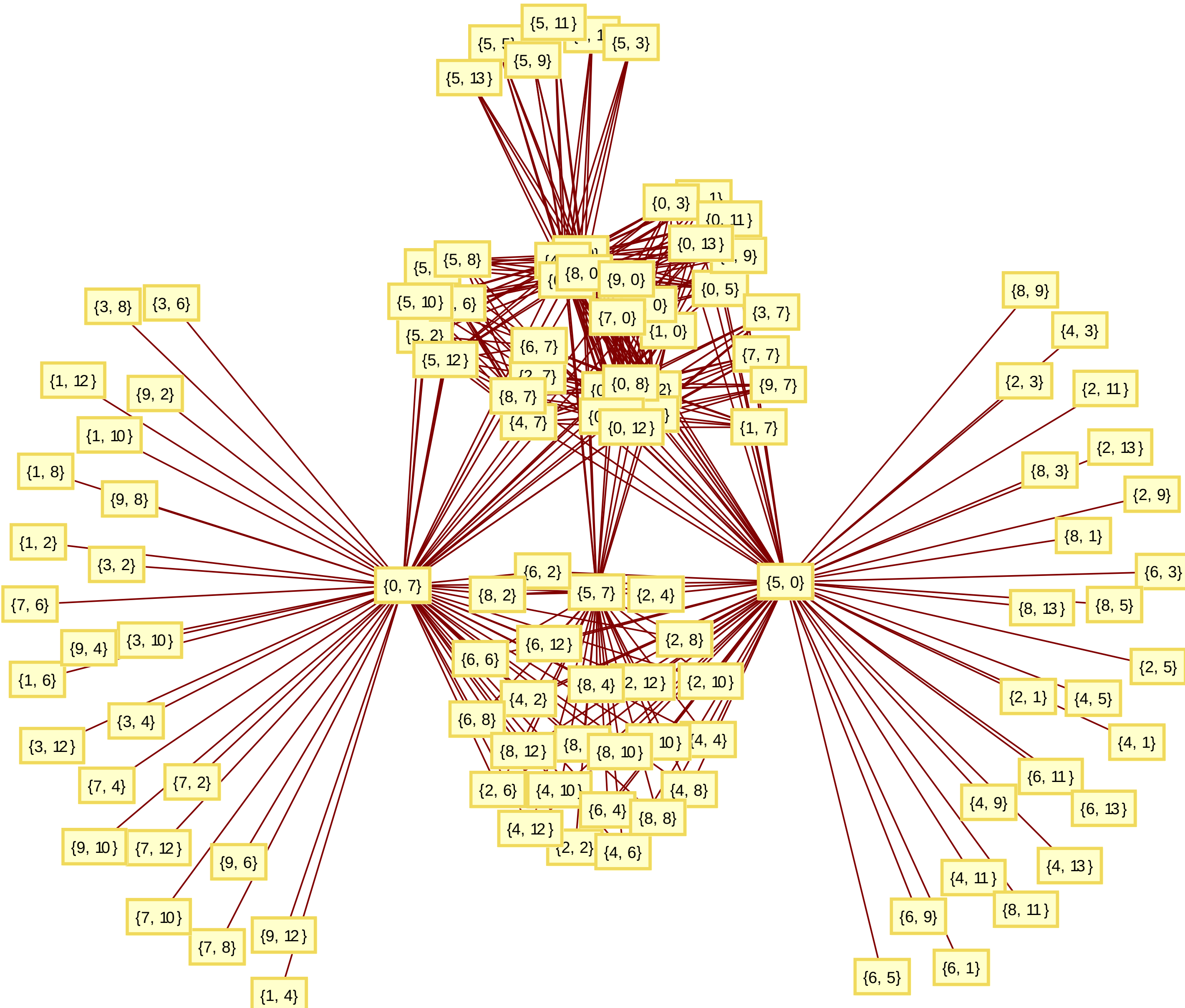

Figure 4: Zero-Divisor Graph of $\boldsymbol{Z}_{10} \times \boldsymbol{Z}_{14}$

Let us analyze one more ring: $\boldsymbol{Z}_{14} \times \boldsymbol{Z}_{22}$, as shown in Figure 5. This graph makes the separate families very clear; we can even count how many families there are. Now we must categorize these families to construct a general form of a zero-divisor graph for $\boldsymbol{Z}_{2p} \times \boldsymbol{Z}_{2q}$.

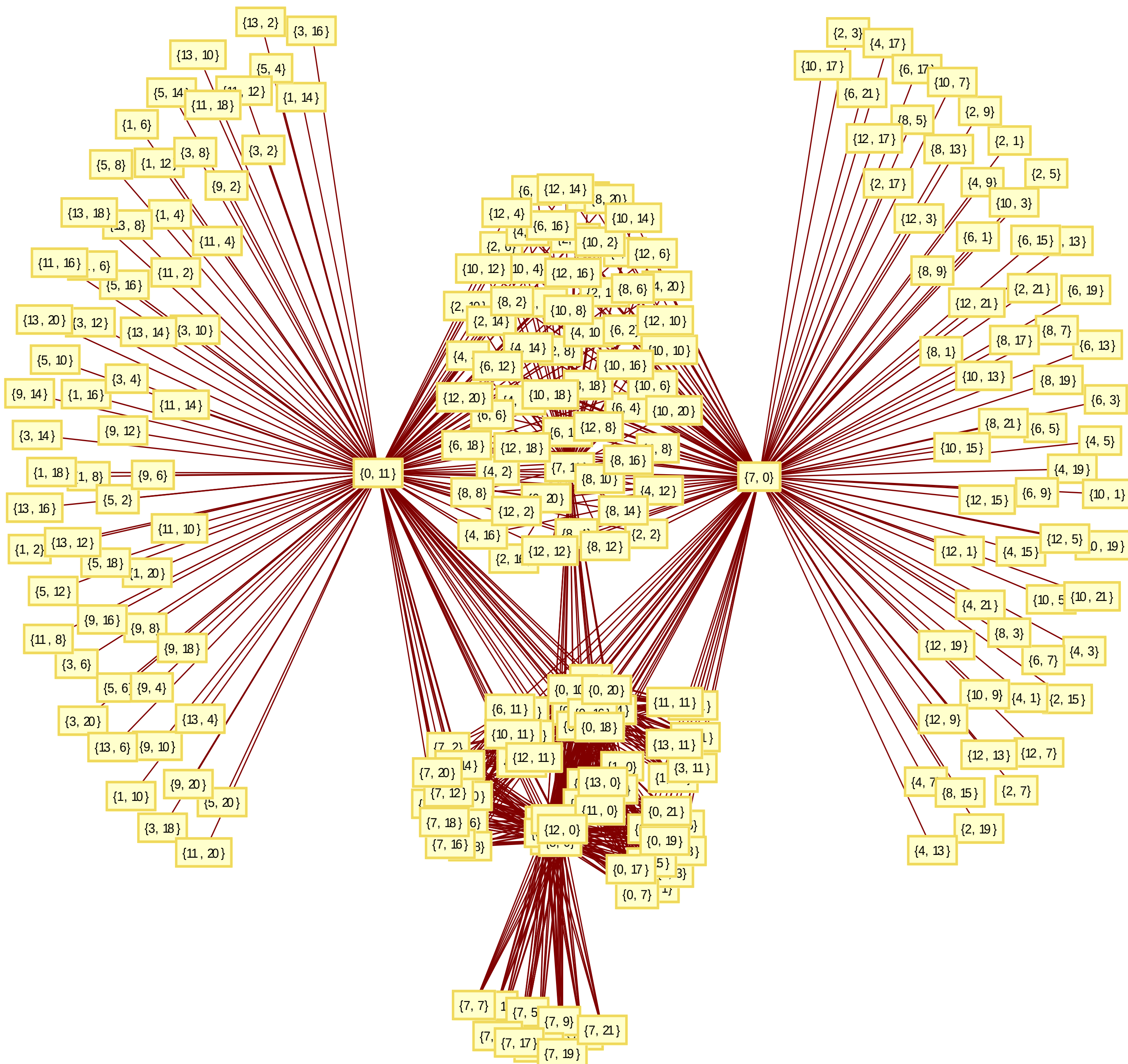

Figure 5: Zero-Divisor Graph of $\boldsymbol{Z}_{14} \times \boldsymbol{Z}_{22}$

Upon investigating these rings, 14 distinct families of zero-divisors were found. Each family contains several elements, and all of the elements in a family will connect to all of the elements in another family if the families are connected with an edge. The 14 families are characterized as follows: $(0, q)$, $(p, 0)$, (p, q), (relative primes to $2p$, evens in $\boldsymbol{Z}_{2q}$), (evens in $\boldsymbol{Z}_{2p}$, relative primes to $2q$), (evens in $\boldsymbol{Z}_{2p}$, evens in $\boldsymbol{Z}_{2q}$), (evens in $\boldsymbol{Z}_{2p}$, 0), (0, evens in $\boldsymbol{Z}_{2q}$), (p, evens in $\boldsymbol{Z}_{2q}$), (evens in $\boldsymbol{Z}_{2p}$, q), (p, relative primes to $2q$), (relative primes to $2p$, q), (relative primes to $2p$, 0), and (0, relative primes to $2q$). For all of the coordinates labeled "evens in $\boldsymbol{Z}_{2p}$" or "evens in $\boldsymbol{Z}_{2q}$," the number 0 is excluded. These families can also be referred to as "multiples of 2 in $\boldsymbol{Z}_{2p}$" or "multiples of 2 in $\boldsymbol{Z}_{2q}$". Figure 2 shows the general form of the zero divisor graph for $\boldsymbol{Z}_{2p} \times \boldsymbol{Z}_{2q}$ rings, where p and q are greater than 2. Note that every element in each rectangle will connect to each element in another rectangle if the rectangles are connected with an edge.

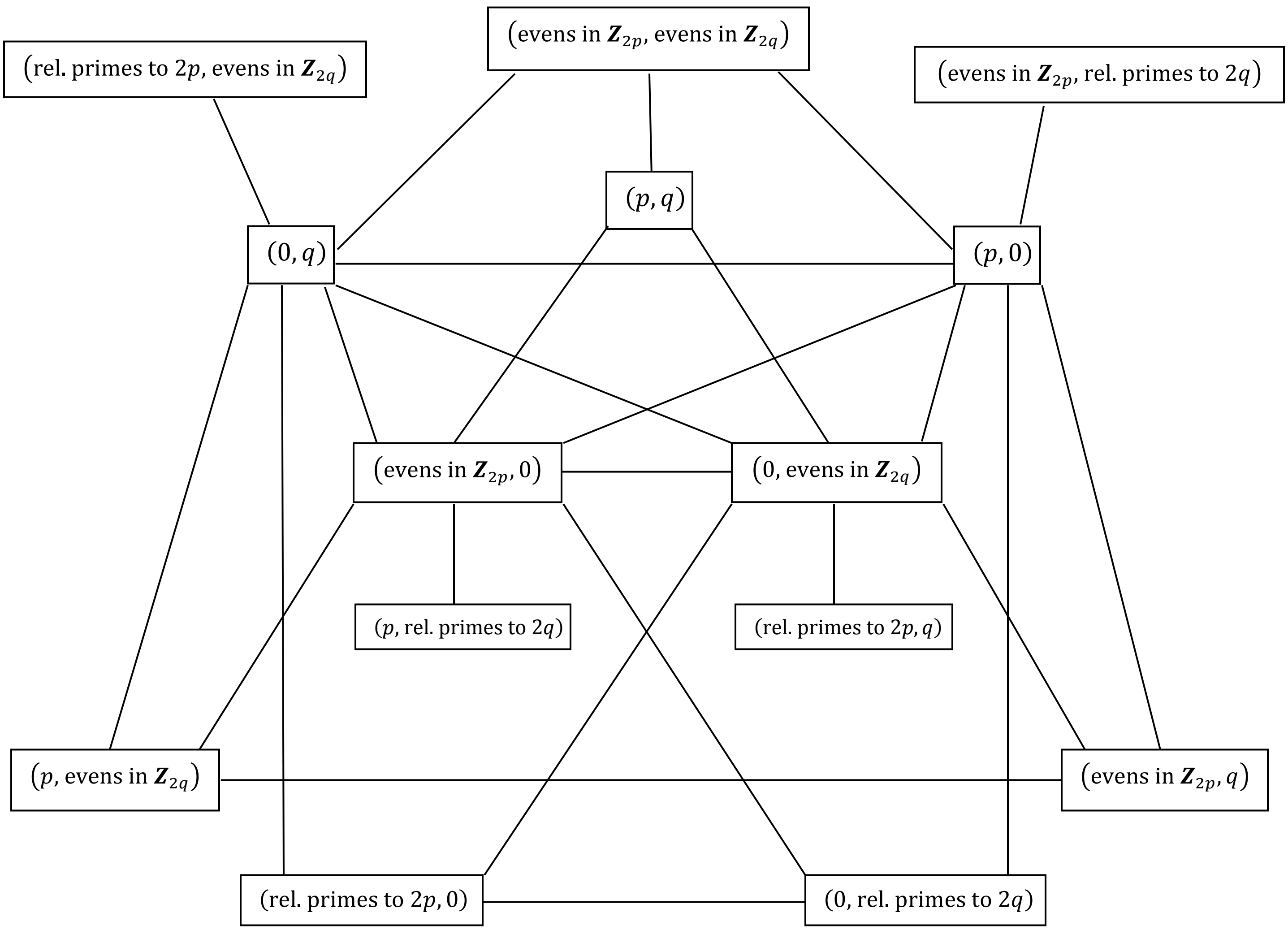

Figure 6: Generalized Form of $\boldsymbol{Z}_{2p} \times \boldsymbol{Z}_{2q}$ Zero-Divisor Graphs

To show that this is the correct graph for all $\boldsymbol{Z}_{2p} \times \boldsymbol{Z}_{2q}$ rings where p and q are prime integers greater than 2, we must prove the existence or nonexistence of each edge. We will do this by proving which coordinates are always zero-divisor pairs.

Proposition 1 Multiples of 2 in $\boldsymbol{Z}_{2p}$ form a zero-divisor pair with p.

Proof. All multiples of 2 in $\boldsymbol{Z}_{2p}$ can be written in the form $2k \bmod 2p$, where $k \in \boldsymbol{Z}$. To see if $2k$ and p form a zero-divisor pair, we must multiply them together to see if the product equals zero. Therefore, we have $2k * p \bmod 2p = k * (2p) \bmod 2p \equiv k * 0 = 0$. Thus, multiples of 2 in $\boldsymbol{Z}_{2p}$ form a zero-divisor pair with p. ■

The same argument can be made for multiples of 2 in $\boldsymbol{Z}_{2q}$ and q. Now that we have proven the existence of a zero-divisor pair, let us prove the nonexistence of a zero-divisor pair.

Proposition 2 Relative primes to $2p$ will not form a zero-divisor pair with p.

Proof. Let m be relatively prime to $2p$. Since they are relatively prime, m must be an odd number since $2p$ is even. Since p is prime and greater than 2, p is odd as well. Because the product of two odd numbers is always odd, the product $m * p$ will be an odd number and can be written as $mp = 2a + 1 \bmod 2p$, where $a \in \boldsymbol{Z}$**.** This can be rewritten as $(2a \bmod 2p) + 1$. The term $(2a \bmod 2p)$ will be an even number which can be written as $2b$, where $b \in \boldsymbol{Z}$. This means that $mp = 2b + 1$, which is odd. If m and p were a zero-divisor pair, mp would equal zero. However, we know mp is an odd number, so it cannot be zero. Therefore, relative primes to $2p$ will not form a zero-divisor pair with p. ■

Now that we have a general structure of a $\boldsymbol{Z}_{2p} \times \boldsymbol{Z}_{2q}$ graph, we can investigate the planarity of these graphs. A graph is said to be "planar" if the edges can be drawn in such a way that none of the edges intersect. If a graph contains a subgraph homeomorphic to $K_{3,3}$, pictured in Figure 7, then the entire graph is not planar.

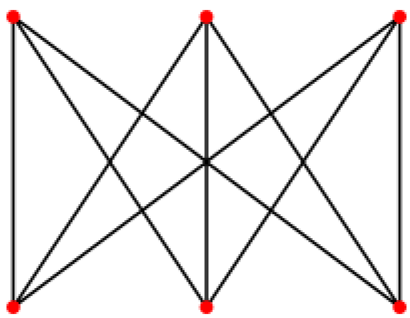

Figure 7: $K_{3,3}$

The graph of $\boldsymbol{Z}_{2p} \times \boldsymbol{Z}_{2q}$ is not planar if certain restrictions are placed on p and q. When both p and q are greater than or equal to 5, the graph will not be planar, because there will be a subgraph homeomorphic to $K_{3,3}$. The families (evens in $\boldsymbol{Z}_{2p}$, 0) and (0, evens in $\boldsymbol{Z}_{2q}$) will have at least three elements. Since all of the elements in (evens in $\boldsymbol{Z}_{2p}$, 0) will connect to all of the elements in (0, evens in $\boldsymbol{Z}_{2q}$), we have a subgroup homeomorphic to $K_{3,3}$. Therefore, when both p and q are greater than or equal to 5, the graph will not be planar.

4 Conclusion and Directions for Further Research

While limited, throughout this research we have generalized the form of all zero-divisor graphs of $\boldsymbol{Z}_{2p} \times \boldsymbol{Z}_{2q}$ rings, where p and q are prime integers greater than 2. All zero-divisor graphs of these rings will have 14 distinct sets, and each zero-divisor will fall into one of these sets. Each element in a set will connect to all of the elements of another set if the sets are connected with an edge on the zero-divisor graph. There are endless possibilities to continue this research, such as looking at different multiples of p and q. For example, one could look at zero-divisor graphs of the form $\boldsymbol{Z}_{3p} \times \boldsymbol{Z}_{3q}$, or $\boldsymbol{Z}_{2p} \times \boldsymbol{Z}_{3q}$, and so on.

References

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