The Fibonacci Sequence Through a Different Lens

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Abstract

While Fibonacci numbers have been the subject of numerous works over hundreds of years, the sequence presents still more opportunities for discovery. This paper introduces a new way of perceiving this well established sequence by looking at Fibonacci relationships in terms of a specific geometric construction based on Fibonacci numbers. We first find that the properties of these shapes imply several Fibonacci relationships by themselves. Moreover, we begin to explore a general form of this construction based on Fibonacci numbers which yields additional Fibonacci relationships, confirms previous observations, and offers more conclusions about Fibonacci numbers for further research. As we work through conclusions previously reached by others, we find that there is yet another path to our known destination, which suggests other related trails for further expansion of our understanding of this enduring sequence.

1 Introduction

The Fibonacci sequence, though centuries old, stands as one of the most profound and dynamic sequences in mathematics. Based on a simple recursive formula, these numbers have applications and relationships which have been of interest to mathematicians, artists and scientists throughout the years. While these numbers are identified as early as the sixth century in Indian poetry [5], the Fibonacci numbers are best known for being part of the work of their 13th century namesake: Fibonacci. Nestled in one of his several mathematical texts, *Liber Abaci*, a problem concerning the growth of a rabbit population introduced this sequence to Western society for the first time. Ironically it has since eclipsed Fibonacci's crucial role in the introduction of Hindu numerals and many of the common mathematical conventions which allowed Western mathematics to move beyond the abacus and the Roman numeral [1]. Nevertheless, this sequence is of great significance by itself, and many volumes have been dedicated to its properties and applications.

This paper will take an alternative approach to the Fibonacci sequence. While not discovering new properties, we offer a different way to perceive familiar concepts and, perhaps with an altered lens, see a new side to a very old sequence. First we review the Fibonacci sequence itself, and its partner concept; the golden ratio. After introducing a couple of key definitions related to these foundational ideas, we will define our main set of circles based on Fibonacci numbers, which allow us to explore the Fibonacci sequence with the circles as our guides. We then broaden these circles into a general form that will show us even more relationships between Fibonacci numbers. As is characteristic of this continually fascinating sequence, we finish by discussing areas of further research brought up by this alternative approach which may yield even more results on this topic.

2 Definitions and Development

To begin a more thorough treatment of this famous sequence, we first review a definition of the sequence itself.

Definition 1 The **Fibonacci Sequence** is the sequence of numbers beginning with $F_0 = 0$ and $F_1 = 1$ where the *n*th term, denoted F_n, is $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

In order to get an idea as to what this sequence looks like, Figure 1 presents the first 10 elements of the Fibonacci sequence. This sequence can continue to an infinite number of terms and interesting patterns do appear as we add more terms. [1]

n	F_n
1	1
2	1
3	2
4	3
5	5
6	8
7	13
8	21
9	34
10	55

Figure 1: The first 10 numbers of the Fibonacci sequence

As we proceed, we also find that the Fibonacci sequence goes hand-in-hand with a concept which comes from Ancient Greece known as the golden ratio. Here we denote the golden ratio as ϕ and define it as follows.

Definition 2 The **Golden Ratio** (ϕ) is the positive solution of $x^2 - x - 1 = 0$ or $x = 1 + \frac{1}{x}$ and is expressed as $\frac{1+\sqrt{5}}{2}$ or approximately 1.618.

Note that this definition can quickly lead us towards a few conclusions about the golden ratio such as the fact that the number $(-\phi)^{-1} = \frac{1-\sqrt{5}}{2}$ is the second solution of the polynomial used in Definition 2, and the equation $\phi = 1 + \frac{1}{\phi}$ is true. One way in which we can see the close relationship between the golden ratio and Fibonacci numbers is by showing that the golden ratio can be expressed in terms of the Fibonacci numbers. Specifically we can write $\phi = \frac{(F_0+F_2)+F_2\sqrt{5}}{2}$ and further show that $\phi^n = \frac{(F_{n-1}+F_{n+1})+F_{n+1}\sqrt{5}}{2}$. The proof of this fact is not within the scope of this paper, but it is sufficient to say that this number is key to our understanding of the Fibonacci numbers.

Another concept that will be useful for an understanding of Fibonacci numbers and the golden ratio is the linear fractional transformation. In general the linear fractional transformation is defined as follows.

Definition 3 A **linear fractional transformation** (LFT) is any function *T* written in the form of

$$T(z) := \frac{az + b}{cz + d}$$

where *a*, *b*, *c*, and *d* are complex numbers with $ad - bc \neq 0$ [3].

While there are many properties of LFT's, the most useful property presently is that $T(z)$ will map circles and lines to circles or lines. More formally, the latter property can be written in terms of $\mathcal{S}$, the set of circles and lines in the complex plane. If $C \in \mathcal{S}$, then $T(C)$ is also an element of $\mathcal{S}$ [3].

With this definition in mind, let us find our own specific LFT. We can break up the general form of a continued fraction with the function $t(z)$ defined as $t(z) = 1 + \frac{1}{z}$. Let *t* be

composed by itself. If we compose t by itself once, we find $t(t(z)) = t^2(z) = 1 + \frac{1}{1+\frac{1}{z}}$. As we repeat this process $n-1$ times to get $t^n(z)$, we find that we can simplify $t^n(z)$ to the function

$$T_n(z) = \frac{F_{n+1}z+F_n}{F_n z+F_{n-1}}.$$

Using induction, we can definitively show that $t^n(z) = T_n(z)$. Thus $t^n(z) = T_n(z)$ is our linear fractional transformation based on Fibonacci numbers.

With these concepts in hand, we can begin to examine a specific construction of circles which involves these concepts. Let π_n be the circle having a center point of $\left({F_{n+1}}/{F_n}, {1}/{2{F_n}^2}\right)$ and a radius of ${1}/{2{F_n}^2}$ when $F_n \neq 0$. In the case where $F_n = 0$, let π_n be

$$\{\pi_n = z: Im(z) = {F_{n+1}}^2\} \cup \infty.$$

Since $F_n = 0$ implies $n = 0$, we can let $\{\pi_0 = z: Im(z) = 1\} \cup \infty$ or the line $Im(z) = 1$. This produces an infinite number of circles, whose first few circles are illustrated below.

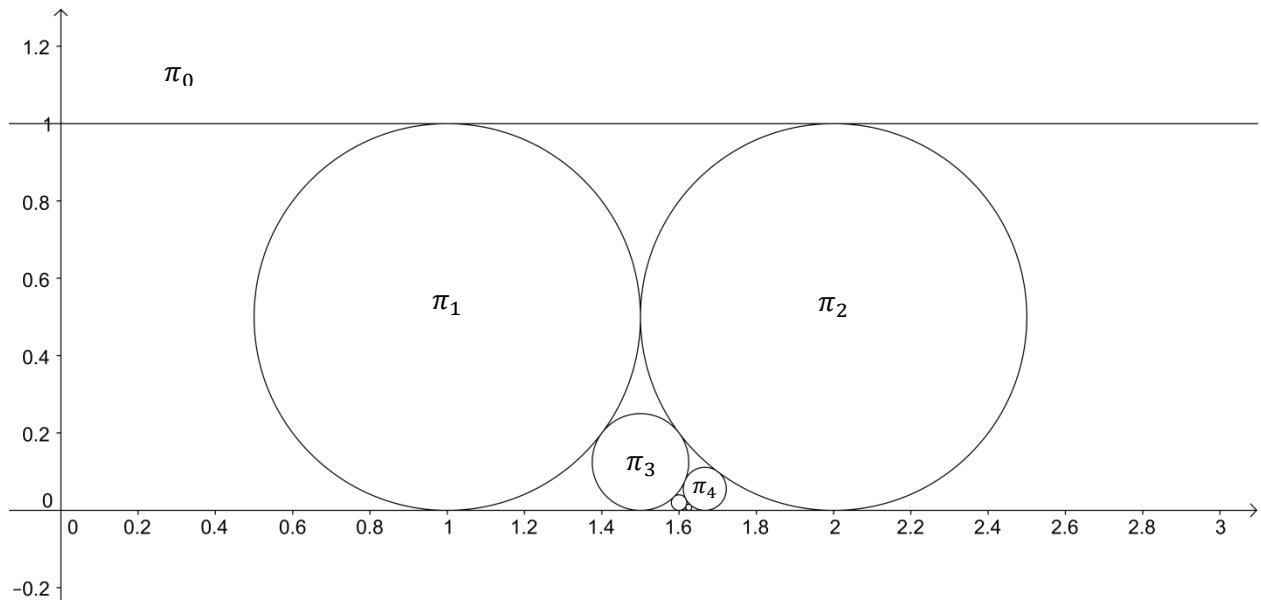

Figure 2: Circles π_0 through π_6

Since these circles are derived from Fibonacci numbers, we can find relationships between Fibonacci numbers based solely on these circles. As it happens we can also learn about these circles with our function $T_n(z)$, since we can recreate these circles by mapping points along the horizontal line to points on the circles. It is from this relationship that we will receive even more conclusions about Fibonacci numbers.

3 Results

The first few Fibonacci relationships we find need only to be determined by the geometry of the circles π_n. However, we will first prove our first relationship by induction based on what we know of Fibonacci numbers so far to verify our conclusion and compare these methods.

Proposition 4 For Fibonacci numbers F_n, ${F_{n+1}}^2 - F_{n+2}F_n = (-1)^n$, when $n \in \mathbb{W}$.

Proof. Because ${F_1}^2 - F_2F_0 = 1 - 0 = (-1)^0$,and ${F_2}^2 - F_3F_1 = 1 - 2 = (-1)^1$ we may assume that ${F_{k+1}}^2 - F_{k+2}F_k = (-1)^k$ for some $k \geq 0$. By Definition 1 we compute that

$$\begin{aligned}
{F_{k+2}}^2 - F_{k+3}F_{k+1} &= (F_{k+1} + F_k)^2 - (F_{k+2}F_{k+1} + {F_{k+1}}^2) \\
&= {F_{k+1}}^2 + 2F_{k+1}F_k + {F_k}^2 - (F_{k+2}F_{k+1} + {F_{k+1}}^2) \\
&= 2F_{k+1}F_k + {F_k}^2 - F_{k+2}F_{k+1} \\
&= 2F_{k+1}F_k + {F_k}^2 - \left({F_{k+1}}^2 + F_{k+1}F_k\right) \\
&= F_{k+1}F_k + {F_k}^2 - {F_{k+1}}^2 \\
&= F_k(F_{k+1} + F_k) - {F_{k+1}}^2 \\
&= (-1)\left({F_{k+1}}^2 - F_kF_{k+2}\right) \\
&= (-1)^{k+1}
\end{aligned}$$

Thus we conclude that ${F_{n+1}}^2 - F_{n+2}F_n = (-1)^n$ for $n \in \mathbb{W}$ by induction. ■

While this relationship is true simply based on the definition of a given Fibonacci number, it is also shown by π_n. Notice that π_n is tangent to both π_{n+1} and π_{n+2}. In general, if two circles with center points (a, b) and (c, d), where points *b* and *d* are positive, are tangent to each other and both are tangent to the x-axis, then we can state that $(a - c)^2 = 4bd$ based on the Pythagorean theorem. For the circles described by π_n this implies both the equations

$$\left({F_{n+1}}/{F_n} - {F_{n+2}}/{F_{n+1}}\right)^2 = {1}/{{F_n}^2{F_{n+1}}^2}, \qquad (1)$$

based on π_n being tangent to π_{n+1} at the primary tangent point, and

$$\left({F_{n+1}}/{F_n} - {F_{n+3}}/{F_{n+2}}\right)^2 = {1}/{{F_n}^2{F_{n+2}}^2},$$

based on π_n being tangent to π_{n+2} at the secondary tangent point. These two equations further lead to the conclusions that

$$({F_{n+1}}^2 - F_{n+2}F_n)^2 = 1 \text{ and } (F_{n+1}F_{n+2} - F_{n+3}F_n)^2 = 1.$$

While this essentially strikes upon the relationship proved in Proposition 4, we are still only left with the conclusion that ${F_{n+1}}^2 - F_{n+2}F_n = \pm 1$. Upon closer examination of the graph, we find that the sign of ± 1 is determined by the difference in x-coordinates between the centers of the tangent circles. If π_n is to the left of π_{n+1}, then we must have a negative sign, since the center of π_n is less than the center of π_{n+1}. If π_n is to the right of π_{n+1}, the opposite is true. We can immediately see from Figure 1, that if π_n is to the left of π_{n+1}, then *n* is even, and that if π_n is to the right of π_{n+1}, *n* is odd. Therefore we can tell that ${F_{n+1}}^2 - F_{n+2}F_n = (-1)^n$. As a bonus, we also tell that $F_{n+1}F_{n+2} - F_{n+3}F_n = (-1)^n$, by similar reasoning.

This result directly contributes to another conclusion we can reach just through π_n. Specifically Equation (1) can lead us to the conclusion that $\left({F_{n+1}}/{F_n}\right) \to \phi$. For a complete proof we will start by observing a key property of $\left({F_{n+1}}/{F_n}\right)$. Recall the function $t(z)$ we previously described as $t(z) = 1 + \frac{1}{z}$. We have already seen that $t^n(z) = T_n(z)$. However as we calculate $t^n(1)$ at various points we can also quickly see that it is calculating ratios of Fibonacci numbers. For example, $t(1) = 1 + \frac{1}{1+1} = \frac{3}{2}$, $t^2(1) = 1 + \frac{1}{1+\frac{1}{1+1}} = \frac{5}{3}$, and $t^3(1) = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+1}}} = \frac{8}{5}$. As we calculate these results we see that each result can be restated in terms of the previous result so that $t^2(1) = 1 + \frac{1}{1+\frac{1}{1+1}} = 1 + \left(\frac{3}{2}\right)^{-1} = \frac{5}{3}$ and $t^3(1) = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+1}}} = 1 + \left(\frac{5}{3}\right)^{-1} = \frac{8}{5}$. This pattern can be generalized to $\frac{F_{n+2}}{F_{n+1}} = 1 + \left(\frac{F_{n+1}}{F_n}\right)^{-1}$ which we can show as

$$\frac{F_{n+2}}{F_{n+1}} = \frac{(F_{n+1} + F_n)}{F_{n+1}} = 1 + \left(\frac{F_{n+1}}{F_n}\right)^{-1} \qquad (2)$$

from Definition 1.

Proposition 5 For Fibonacci numbers F_n, $\left({F_{n+1}}/{F_n}\right) \to \phi$, when $n \in \mathbb{N}$.

Proof. First, we can show that Equation (1) implies that $\left({F_{n+1}}/{F_n}\right)$ is contractive. By the definition of a contractive sequence, a contractive sequence (x_n) will satisfy $|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n|$ where $0 < C < 1$ [4]. By Equation (1) we can state that $\left|{F_{n+2}}/{F_{n+1}} - {F_{n+1}}/{F_n}\right| = {1}/{F_n F_{n+1}}$. As a result of this equality, our proof can now rest on the relationship between $F_n F_{n+1}$ and $F_{n+1} F_{n+2}$. By the Definition 1 we calculate

$$F_{n+1} F_{n+2} = {F_{n+1}}^2 + F_{n+1} F_n \geq 2 F_{n+1} F_n,$$

where *n* is a whole number, and by implication conclude that ${1}/{F_{n+1} F_{n+2}} \leq {1}/{2 F_n F_{n+1}}$. Thus

$$\left|{F_{n+3}}/{F_{n+2}} - {F_{n+2}}/{F_{n+1}}\right| = {1}/{F_{n+1} F_{n+2}} \leq {1}/{2 F_n F_{n+1}} = {1}/{2}\left|{F_{n+2}}/{F_{n+1}} - {F_{n+1}}/{F_n}\right|$$

and we conclude $\left({F_{n+1}}/{F_n}\right)$ is contractive by the definition of a contractive sequence. This conclusion then implies that $\left({F_{n+1}}/{F_n}\right)$ is a convergent sequence.

To determine that $\left({F_{n+1}}/{F_n}\right) \to \phi$, we can pass to the limit using Equation (2). Since we know that $\left({F_{n+1}}/{F_n}\right) \to x$, we also that $\left({F_{n+2}}/{F_{n+1}}\right) \to x$. By Equation (2), this implies

$$x = 1 + (x)^{-1}$$

which by Definition 2 means $x = \phi$. Note that since $F_n \geq 0$ and ${F_{n+1}}/{F_n} > 0$ while *n* is a whole number, x must be positive. This fact eliminates the second option of Definition 2 that $x = (-\phi)^{-1}$. Therefore $\left({F_{n+1}}/{F_n}\right) \to \phi$ for $n \in \mathbb{N}$. ■

At this point, we can proceed to examine the circles through $T_n(z)$. As mentioned before, the function $T_n(z)$, being a linear fractional transformation, will map all points on a line or circle to a line or circle. In this case, we take points along the line $Im(z) = 1$, map them to $T_n(z)$, map them again to $T_n(T_n(z))$, and so on, to produce the same circles as π_n. However this process only recreates π_n exactly when *n* is even. When *n* is odd, then the circles of $T_n(z)$ alternate on the positive and negative sides of the x-axis while retaining the same x-coordinate and radius. So for example, Figures 3 and 4 show the circles mapped from the line $Im(z) = 1$ by $T_1(z)$ and $T_2(z)$.

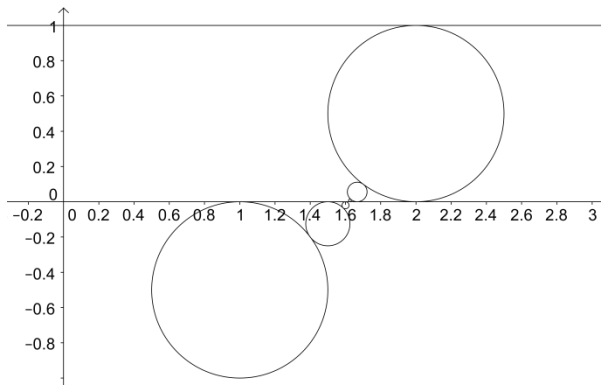

Figure 3: The first 6 circles from the line $Im(z) = 1$ using $T_1(z)$

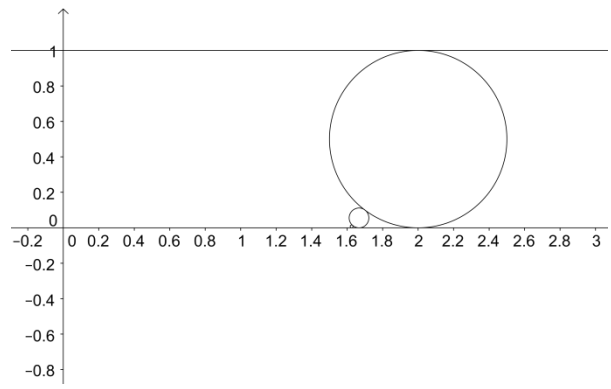

Figure 4: The first 4 circles from the line $Im(z) = 1$ using $T_2(z)$

In fact, we find not only that this principle holds true, but that if we number the circles according to the number of times $T_1(z)$ was composed from $Im(z) = 1$, $T_n(z)$ should map the mth circle to the $m + n$th circle, with the sign of the y-coordinate dependent on $(-1)^n$.

Before we explore the implications of this map, let us establish some Fibonacci principles we will need later.

Proposition 6 For Fibonacci numbers F_n, $F_{n+x} = F_{n+1}F_x + F_nF_{x-1}$, when $n \in \mathbb{W}$ and $x \in \mathbb{N}$.

Proof. Since $F_{n+1} = F_{n+1}F_1 + F_nF_0 = F_{n+1}$, $F_{n+2} = F_{n+1}F_2 + F_nF_1 = F_{n+2}$, and $F_{n+3} = F_{n+1}F_3 + F_nF_2 = F_{n+1} + (F_{n+1} + F_n) = F_{n+1} + F_{n+2} = F_{n+3}$, let us assume that $F_{n+i} = F_{n+1}F_i + F_nF_{i-1}$ for all $1 \leq i \leq k, k \in \mathbb{N}$. Then we compute

$$\begin{aligned} F_{n+k+1} &= F_{n+k} + F_{n+k-1} \\ &= F_{n+1}F_k + F_nF_{k-1} + F_{n+1}F_{k-1} + F_nF_{k-2} \\ &= F_{n+1}(F_k + F_{k-1}) + F_n(F_{k-1} + F_{k-2}) \\ &= F_{n+1}F_{k+1} + F_nF_k. \end{aligned}$$

Thus by strong induction, $F_{n+x} = F_{n+1}F_x + F_nF_{x-1}$ for $n \in \mathbb{W}$ and $x \in \mathbb{N}$. ■

Proposition 7 For Fibonacci numbers F_n, $F_{n+x}F_{n-1} - F_{n+x-1}F_n = (-1)^n F_x$, when $n \in \mathbb{W}$ and $x \in \mathbb{N}$.

Proof. Since we have already shown $F_{n+1}F_{n-1} - {F_n}^2 = (-1)^n F_1$ in Proposition 5, let us assume $F_{n+i}F_{n-1} - F_{n+i-1}F_n = (-1)^n F_i$ for all $1 \le i \le k, k \in \mathbb{N}$. Then we compute

$$(-1)^n F_{k+1} = (-1)^n F_k + (-1)^n F_{k-1}$$

$$= F_{n+k}F_{n-1} - F_{n+k-1}F_n + F_{n+k-1}F_{n-1} - F_{n+k-2}F_n$$

$$= F_{n-1}(F_{n+k} + F_{n+k-1}) - F_n(F_{n+k-1} + F_{n+k-2})$$

$$= F_{n-1}F_{n+k+1} - F_nF_{n+k}$$

Thus by strong induction, $F_{n+x}F_{n-1} - F_{n+x-1}F_n = (-1)^n F_x$ for $n \in \mathbb{W}$ and $x \in \mathbb{N}$. ■

Now we are ready to observe our first fact about π_n based on $T_n(z)$. Namely we can determine at what point π_n is tangent to π_{n+1} or the primary tangent point of π_n. While it is simple enough to observe that $\pi_1$is tangent to π_2 at $\left({}^3/_2\ , {}^1/_2\right)$, it takes a little more time to determine all other primary tangent points. We could muddle through Fibonacci relationships using the center points and radii, but here it is instead preferable to use $T_n(z)$. While almost all other points on these circles shift around in more complicated ways when mapped using $T_n(z)$, the one kind of point that will always map reliably is the point where the circles are tangent. For example, we can see in Figure 4, the primary tangent point of π_n maps to the primary tangent point of π_{n+2} using $T_2(z)$. We could also show a similar result for $T_1(z)$ where the primary tangent point maps to the next primary tangent point, if we modify for the alternating signs of the circles.

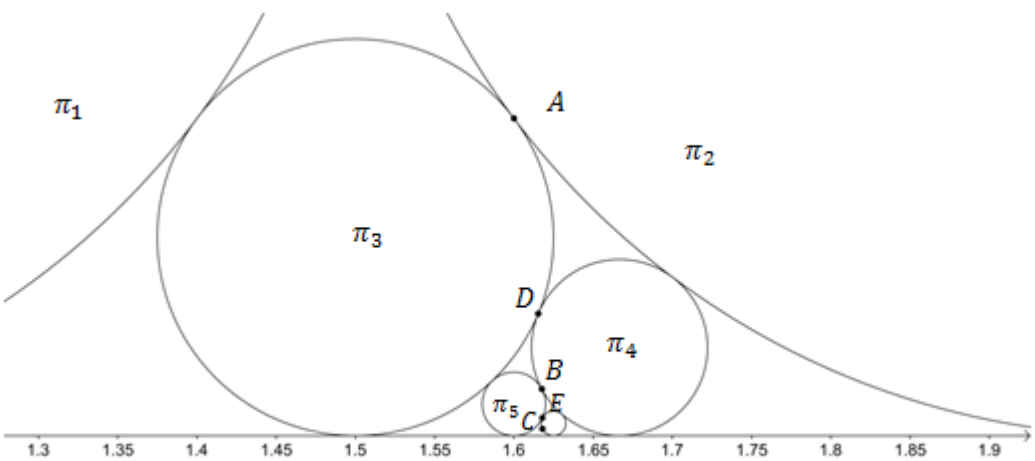

Figure 5: Using $T_2(z)$, A maps to B maps to C, while D maps to E

If we use $T_2(z)$ to find the points where the circles are tangent, we find that, with the first few circles, the primary point of π_n is at $\left({}^{F_{2n+2}}/_{F_{2n+1}}, {}^1/_{F_{2n+1}}\right)$. Geometrically, this solution is consistent with the centers and radii of the tangent circles since it can be found on both π_n and π_{n+1}. While details are not included here, we found this is to be true using the distance formula

for each circle which required Proposition 8 and the Fibonacci relationship $F_{2n+1} = {F_{n+1}}^2 + {F_n}^2$, found in Lehmann or derived from Proposition 7. Thus we can conclude that $\left({F_{2n+2}}/{F_{2n+1}}, {1}/{F_{2n+1}}\right)$ is the primary tangent point of π_n.

A second result of this relationship is that a given primary tangent point will map to the primary tangent point for any $T_n(z)$. To show that this is true, a couple more propositions will be required.

Proposition 8 For Fibonacci numbers F_n, $F_{2x+2n+2}F_{2n+1} - F_{x+2n+2}F_{x+2n+1} = F_{x+1}F_x$, when $n, x \in \mathbb{W}$.

Proof. Since $F_{0+2n+2}F_{2n+1} - F_{0+2n+2}F_{0+2n+1} = F_{2n+2}F_{2n+1} - F_{2n+2}F_{2n+1} = F_1F_0$, and $F_{2+2n+2}F_{2n+1} - F_{1+2n+2}F_{1+2n+1} = F_{2n+4}F_{2n+1} - F_{2n+3}F_{2n+2} = (-1)^{2n+2}F_2 = 1 = F_2F_1$by Proposition 8, let us assume $F_{2k+2n+2}F_{2k+1} - F_{k+2n+2}F_{k+2n+1} = F_{k+1}F_k$ for some $k \in \mathbb{N}$. We then compute

$$F_{2k+2n+4}F_{2k+1} - F_{k+2n+3}F_{k+2n+2} =$$

$$F_{2k+2n+4}F_{2n+1} - F_{2k+2n+3}F_{2n+2} + F_{2k+2n+3}F_{2n+2} - F_{k+2n+3}F_{k+2n+2} =$$

$$(-1)^{2n+2}F_{2k+2} + F_{2k+2n+3}F_{2n+2} - F_{k+2n+3}F_{k+2n+2},$$

by Proposition 7. We can further compute

$$(-1)^{2n+2}F_{2k+2} + F_{(k+2n+3)+k}F_{2n+2} - F_{k+2n+3}F_{k+(2n+2)} =$$

$$F_{2k+2} + (F_{k+1}F_{k+2n+3} + F_kF_{k+2n+2})F_{2n+2} - F_{k+2n+3}(F_{k+1}F_{2n+2} + F_kF_{2n+1})$$

by Proposition 6. We then simplify this statement and compute

$$F_{2k+2} + F_k(F_{2n+2}F_{k+2n+2} - F_{2n+1}F_{k+2n+3}) =$$

$$F_{2k+2} + -F_k(-1)^{2n+2}F_{k+1} =$$

$$F_{2k+2} - F_kF_{k+1} = F_{k+2}F_{k+1}$$

using Proposition 7 then Proposition 6. Thus $F_{2x+2n+2}F_{2n+1} - F_{x+2n+2}F_{x+2n+1} = F_{x+1}F_x$ for $n, x \in \mathbb{W}$ by induction. ■

Proposition 9 For Fibonacci numbers F_n, ${F_{2x+n+1}}^2 + {F_n}^2 = F_{2x+2n+1}F_{2x+1}$, when $n, x \in \mathbb{W}$.

Proof. Using Proposition 7 we can first state

$${F_{2x+n+1}}^2 + {F_n}^2 = \left(F_{2x+n+2}F_{2x+n} - (-1)^{2k+n+1}\right) + (F_{n+1}F_{n-1} - (-1)^n).$$

We further compute

$$F_{2x+n+2}F_{2x+n} - (-1)^{2k+n+1} + F_{n+1}F_{n-1} - (-1)^n = F_{2x+n+2}F_{2x+n} + F_{n+1}F_{n-1} - (-1)^n(-1+1)$$

$$= F_{2x+n+2}F_{2x+n} + F_{n+1}F_{n-1}$$

$$= F_{2x+n+2}F_{2x+n} + (-F_{n+1}F_{n-1}(-1)^{2k+1})$$

$$= F_{2x+n+2}F_{2x+n} + (-F_{n-1}(F_{2x+n+2}F_{2x} - F_{2x+n+1}F_{2x+1}))$$

$$= F_{2x+n+2}F_{2x+n} + (F_{n-1}F_{2x+n+1}F_{2x+1} - F_{n-1}F_{2x+n+2}F_{2x}))$$

$$= F_{2x+n+2}F_{2x+n} + (F_nF_{2x+n+2} + F_{n-1}F_{2x+n+1})F_{2x+1} - (F_nF_{2x+1} + F_{n-1}F_{2x})F_{2x+n+2}$$

$$= F_{2x+n+2}F_{2x+n} + (F_{2x+2n+1})F_{2x+1} - (F_{2x+n})F_{2x+n+2}$$

$$= F_{2x+2n+1}F_{2x+1}$$

using Proposition 6. ■

Proposition 10 For Fibonacci numbers F_n and the linear fractional transformation, $T_n(z)$, $T_n(z)$ will map the primary tangent point of π_k to the primary tangent point of π_{n+k} where the imaginary component of the center of π_{n+k} and π_{n+k+1} has the sign $(-1)^n$, or

$$T_n\left({F_{2k+2}}/{F_{2k+1}}, {1}/{F_{2k+1}}\right) = \left({F_{2(n+k)+2}}/{F_{2(n+k)+1}}, {(-1)^n}/{F_{2(n+k)+1}}\right).$$

Proof. Based on the definition of $T_n(z)$, we compute

$$T_n\left({F_{2k+2}}/{F_{2k+1}} + {i}/{F_{2k+1}}\right) = \frac{F_{n+1}\left({F_{2k+2}}/{F_{2k+1}} + {i}/{F_{2k+1}}\right) + F_n}{F_n\left({F_{2k+2}}/{F_{2k+1}} + {i}/{F_{2k+1}}\right) + F_{n-1}}$$

$$= \frac{\frac{F_{n+1}F_{2k+2} + F_nF_{2k+1} + F_{n+1}i}{F_{2k+1}}}{\frac{F_nF_{2k+2} + F_{n-1}F_{2k+1} + F_ni}{F_{2k+1}}}$$

$$= \frac{F_{n+1}F_{2k+2} + F_nF_{2k+1} + F_{n+1}i}{F_nF_{2k+2} + F_{n-1}F_{2k+1} + F_ni}.$$

Using Proposition 6 we can simplify this statement to

$$T_n\left({F_{2k+2}}/{F_{2k+1}} + {i}/{F_{2k+1}}\right) = \frac{F_{2k+n+2} + F_{n+1}i}{F_{2k+n+1} + F_ni}.$$

We can further compute

$$T_n\left({F_{2k+2}}/{F_{2k+1}} + {i}/{F_{2k+1}}\right) = \frac{(F_{2k+n+2} + F_{n+1}i)(F_{2k+n+1} - F_ni)}{{F_{2k+n+1}}^2 + {F_n}^2}$$

$$= \frac{F_{2k+n+2}F_{2k+n+1} + F_{n+1}F_n + (F_{2k+n+1}F_{n+1} - F_{2k+n+2}F_n)i}{{F_{2k+n+1}}^2 + {F_n}^2}$$

and use Proposition 7 to simplify to

$$= \frac{F_{2k+n+2}F_{2k+n+1} + F_{n+1}F_n + ((-1)^nF_{2k+1})i}{{F_{2k+n+1}}^2 + {F_n}^2}.$$

At this point we must show that each term in this result can be written as a product of F_{2k+1} and then simplify to our desired conclusion. Using Propositions 9 and 10, we can do so. Therefore we conclude

$$\frac{F_{2k+n+2}F_{2k+n+1} + F_{n+1}F_n + ((-1)^n F_{2k+1})i}{{F_{2k+n+1}}^2 + {F_n}^2} = \frac{F_{2k+2n+2}F_{2k+1} + ((-1)^n F_{2k+1})i}{F_{2k+2n+1}F_{2k+1}}$$

$$= \frac{F_{2k+2n+2} + (-1)^n i}{F_{2k+2n+1}} = \frac{F_{2k+2n+2}}{F_{2k+2n+1}} + \frac{(-1)^n i}{F_{2k+2n+1}}.$$

Thus $T_n(z)$ maps the primary tangent point of π_k to the primary tangent point of π_{n+k} after adjusting the imaginary component. ■

Now that we have examined π_n through $T_n(z)$ in detail, we can see what other results we can find from $T_n(z)$. Remember π_n could be found with $T_n(z)$ when we map $Im(z) = 1$. What does $T_n(z)$ look like when we map other lines?

When mapping points from various lines, most cases were not as clean as our starting case. There were certain characteristics that held through the cases which we originally found using $Im(z) = 1$, such as the stability of tangent points and the importance of the golden ratio. Indeed since the golden ratio is a fixed point of $T_n(z)$, the golden ratio was an anchor for every mapping. However one case in particular offered another Fibonacci relationship which was not readily apparent in our first case.

Where $Im(z) = 1$ demonstrated the sequence limit of Fibonacci numbers, the vertical line $Re(z) = 2$ demonstrates a series limit of the Fibonacci numbers. When we map $Re(z) = 2$, we find that it produces the circles as illustrated in Figure 5.

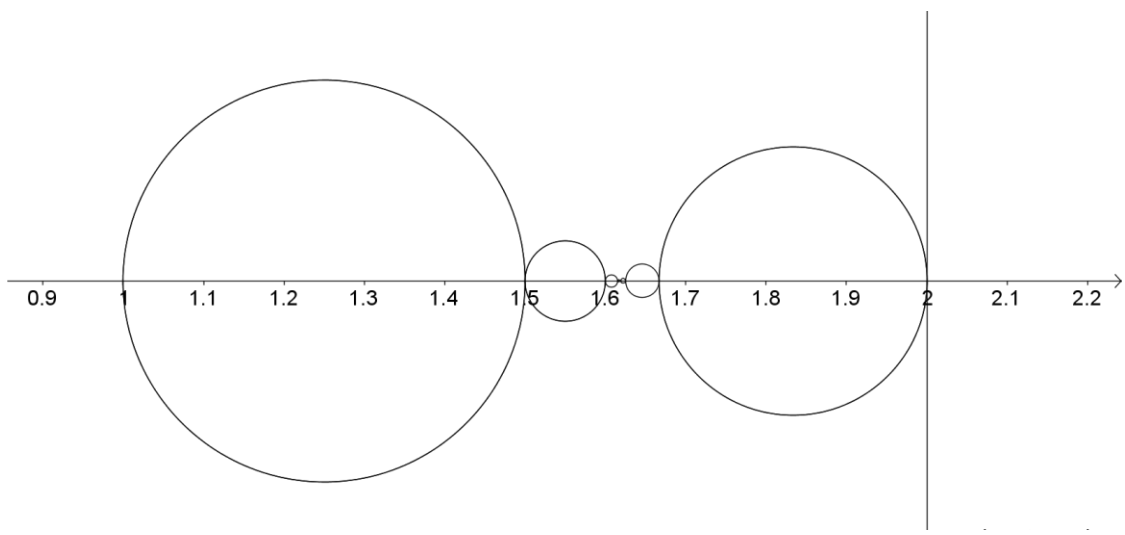

Figure 6: The first 8 circles mapped from $Re(z) = 2$

Similar to the $Im(z) = 1$ map, the original line maps to the largest circle first, which then maps to the next largest, and so on. The circles also progressively move closer and closer to the golden ratio and alternate sides of the golden ratio. Yet in this case, the center of the *n*th circle, starting from the leftmost circle as 1, is at

$$\left(\frac{F_{2n+1}+3F_{n+1}F_n}{2F_{n+2}F_n}, 0\right),$$

and each circle has a radius of $\frac{1}{2F_{n+2}F_n}$. Essentially this map takes $\left({}^{F_{n+1}}/_{F_n}, 0\right)$ as the edges of the circles instead of the center point. The difference now is that the circles span the distance between ${}^{F_{n+1}}/_{F_n}$ and ${}^{F_{n+3}}/_{F_{n+2}}$, which is the diameter of each circle: $\frac{1}{F_{n+2}F_n}$. The result is that the sum of the diameters of circles where *n* is odd, which can also be written as $\sum \frac{1}{F_{2n+1}F_{2n-1}}$, is equal to $\phi - 1$, or ϕ^{-1}. To confirm this conclusion we will take the limit of the partial sums of the diameters of the circles.

Proposition 11 For Fibonacci numbers F_n, $\sum \frac{1}{F_{2n+1}F_{2n-1}} = \phi^{-1}$.

Proof. To determine what the partial sum generally is, we will look at the first few cases.

We compute that

$$s_1 = {}^1/_2$$

$$s_2 = {}^1/_2 + {}^1/_{10} = {}^3/_5$$

$$s_3 = {}^3/_5 + {}^1/_{65} = {}^8/_{13}.$$

Once again we find the Fibonacci numbers present and generalize that

$$s_k = {}^{F_{2k}}/_{F_{2k+1}}.$$

However to truly know that this is the partial sum, we can use induction. We already have shown the first few examples, so let us assume that $s_j = {}^{F_{2j}}\!\big/_{F_{2j+1}}$ for some $j \in \mathbb{N}$. Then we find that

$$s_{j+1} = s_j + \frac{1}{F_{2j+3}F_{2j+1}}$$

$$= \frac{F_{2j}}{F_{2j+1}} + \frac{1}{F_{2j+3}F_{2j+1}}$$

$$= \frac{F_{2j+3}F_{2j}+1}{F_{2j+3}F_{2j+1}}$$

$$= \frac{(F_{2j+3}F_{2j}+(-1)^{2j})}{F_{2j+3}F_{2j+1}}$$

$$= \frac{F_{2j+2}F_{2j+1}}{F_{2j+3}F_{2j+1}}$$

$$= \frac{F_{2j+2}}{F_{2j+3}}.$$

using our extra conclusion of our first relationship. Therefore, since $s_j = {F_{2j}}/{F_{2j+1}}$ implies $s_{j+1} = {F_{2j+2}}/{F_{2j+3}}$, we conclude that $s_k = {F_{2k}}/{F_{2k+1}}$ for $k \in \mathbb{N}$.

Now we only need to observe that Proposition 5 implies that $\left({F_n}/{F_{n+1}}\right) \to \phi^{-1}$, and that $\left({F_{2k}}/{F_{2k+1}}\right)$ is a subsequence of $\left({F_n}/{F_{n+1}}\right)$. Therefore $\left({F_{2k}}/{F_{2k+1}}\right) \to \phi^{-1}$ and $\sum \frac{1}{F_{2n+1}F_{2n-1}} = \phi^{-1}$. ■

4 Conclusion and Directions for Further Research

While the Fibonacci sequence and many of its relationships are previously known, we have demonstrated another way to explore Fibonacci numbers. Based on the geometry of circles which use Fibonacci numbers, we can effectively detect and prove Fibonacci relationships. Furthermore, the map $T_n(z)$ allows us to gain a more precise understanding of these circles and find variations. Since all of these variants are based on Fibonacci numbers, there may easily be more relationships to be found working with them. However during the course of this exercise, certain other topics arose which may be of use for further research.

In the present paper we showed that the primary tangent points of π_n are stable in $T_n(z)$. However we excluded a complete proof of whether or not the secondary tangent points were stable. Thus far we used a similar approach understanding both the secondary tangent points as the primary tangent points. When we did so, it quickly became clear that the points were not directly based on Fibonacci numbers. However, if we define a sequence based on the first few secondary tangent points, we find a Fibonacci-like sequence which we will refer to as a *J*-sequence. This sequence is defined in Appendix A, which includes some initial relationships with Fibonacci numbers based on the definition. The *J*-sequence may be part of some larger sequence, but for the moment, we have sought to understand it only in terms of the numbers found using $T_n(z)$. Presently we do not have a proof showing that the secondary tangent point $\left(\frac{J_{2n+2}}{J_{2n+1}}, \frac{1}{J_{2n+1}}\right)$ is stable but we propose that it is stable. Perhaps in the course of such a proof, new relationships will be found between these sequences that may shed more light on both the *J*-sequence and the Fibonacci sequence.

At the same time, in searching for useful variations based on $T_n(z)$, one mysterious phenomenon repeatedly occurs. As we adjust various diagonal lines, one of the circles becomes so large that it straightens into a line that intersects with the original line. In general it seems that this happens when one of the circles, or the original line, goes through the origin. Appendix B shows a progression of this phenomenon based on the rotation of a single line and the first four circles. However what is interesting is that in each case, when a circle comes to intersect with the original line, it also intersects with $x = .5$. At this point, we are not aware of the significance of this phenomenon. It may relate to the importance of .5 being the exact center of ϕ and $(-\phi)^{-1}$ when $T_n(z)$ is used with ${T_n}^{-1}(z)$, but it is still unclear.

At the same time, matrices may also be applied to these circles to learn more about the Fibonacci numbers and $T_n(z)$. Matrices in themselves are already an incredibly helpful way of understanding Fibonacci sequences. For example, the matrix $M_n = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$ generates Fibonacci numbers when raised to a power, and its determinant leads to Proposition 5; among other conclusions. The matrix M_n can also be understood as just one case of a much larger general structure of sequence generation [5]. In addition to whatever Fibonacci relationships are a result of the matrix itself, the manner in which M_n, or similar matrices affect π_n or $T_n(z)$ is another exciting direction for investigation.

Acknowledgement. The graphs were created using Geogebra, a very useful program available on the Geogebra website.

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Appendix A

Define the *J*-sequence as the sequence where $J_0 = 0$ and $J_1 = 1$ and $J_n = J_{n-1} + J_{n-2} + 2k$ where $k = 0$ if n is even and $k = (-1)^{(n+1/2)}$ if n is odd when $n \geq 2$. This definition produces the following sequence.

n	1	2	3	4	5	6	7	8	9	10	11	12
J_n	1	2	5	7	10	17	29	46	73	119	194	313
$2k$			2	0	-2	0	2	0	-2	0	2	0

Figure 7: The first 12 terms using the definition of the *J*-sequence

At this point we are aware of the following relationships based on this definition of the sequence when *n* is a natural number based on Fibonacci relationships offered by Vajda.

- $3{F_{n+2}}^2 = J_{2n+2} + J_{2n+1}$
- $J_{2n+1} = 3{F_{n+1}}^2 + 2(-1)^{n-1} = {F_{n+2}}^2 + {F_n}^2$
- $F_{n+1}J_{2n+1} - F_nJ_{2n+2} = F_{n+2}(-1)^n$
- $J_{2n+1} - 2{F_n}^2 = F_{n+1}(F_{n+2} + F_n)$

Appendix B

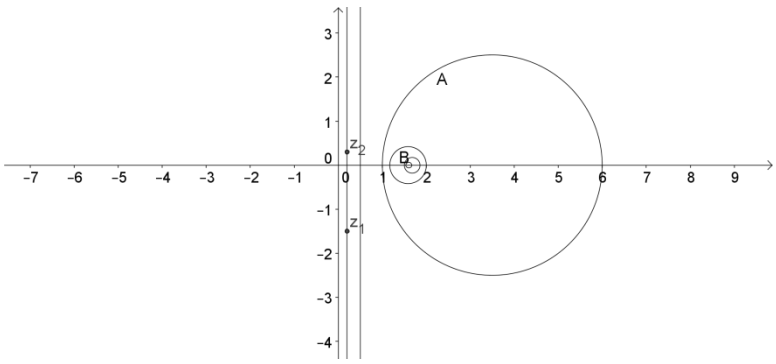

Figure 8: The first 4 circles from $T_1(z)$ starting from $Re(z) = .3$, the line formed by z_1 and z_2. The line $Re(z) = .3$ maps to A then B, C, and D. Circles C and D are the two smaller circles within B. The second vertical line is $Re(z) = .5$

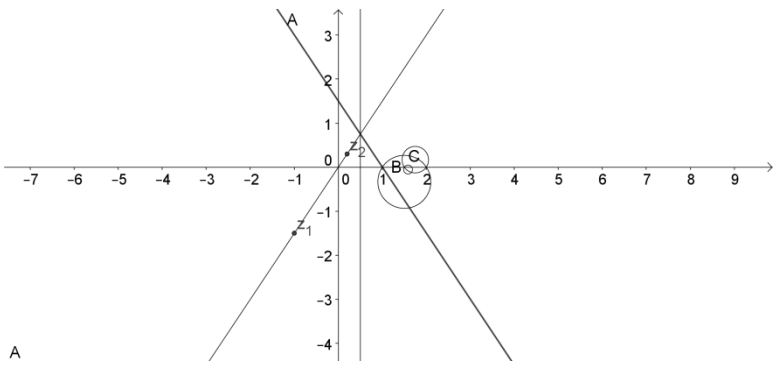

Figure 9: As z_1 moves to the left so that the original lines goes through the origin, A becomes infinitely large. The intersection of A and the original line is on $Re(z) = .5$ as shown.

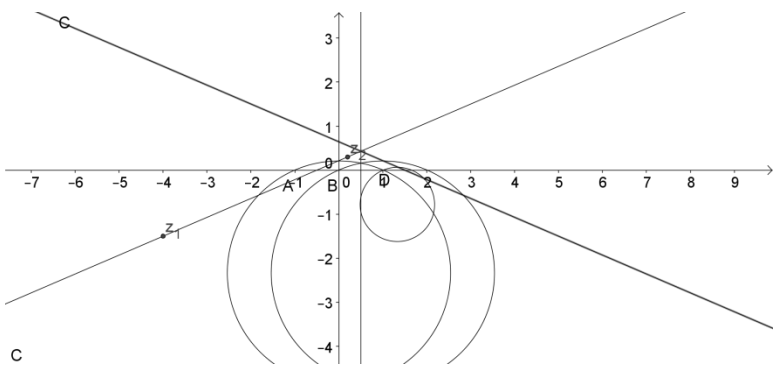

Figure 10: As z_1 moves further to the left, B now goes through the origin, forcing C to become infinitely large. The intersection of C and the original line is also on $Re(z) = .5$.

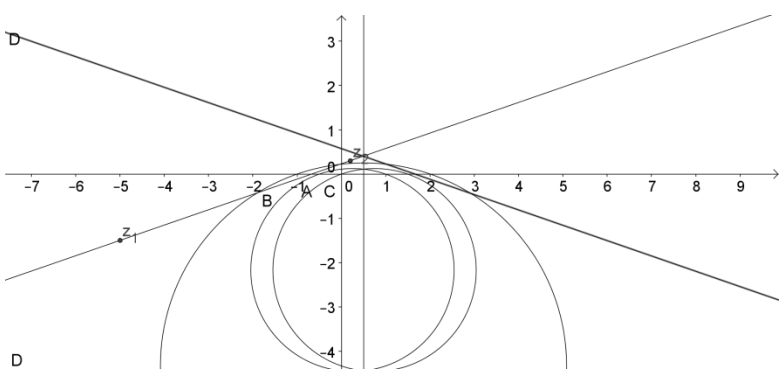

Figure 11: As z_1 moves even further to the left, C goes through the origin, forcing D to become infinitely large. The intersection of D and the original line again is on $Re(z) = .5$.

Note: The Geogebra file which produced these images is available upon request.