The Problem with Baseball Hats

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Abstract

A young baseball player stacks n baseball hats by each door to his home. Every time he leaves the house to go to practice, he grabs a hat from the stack by the door he exits; then when he returns to his home after practice, he leaves his hat on the stack by the door he enters. In our problem we consider how many times, on average, the young baseball player will go out to practice and back into his house before his stack of baseball hats by the door he exits runs out. We will begin with an examination of two doors that start out with n hats by each to determine a formula that calculates how many cycles the boy will run through before he goes to grab a hat as he leaves the house, but instead finds an empty stack. We will then broaden our focus as we start to consider the implications other such nuances and alterations might bring to the problem, looking at what happens when we evaluate the variance surrounding our average (that is, the greatest and least number of times before he runs out of hats), add additional doors, introduce the very likely probability that the boy could lose a hat (or come back with extra hats!), and so forth. Thus we will see the problem with baseball hats as we seek to find the solution—though perhaps not quite the solution an actual baseball player (or his mom) might be looking for.

1 Introduction

"Mom! Where's my hat!" Chances are you have either heard or said this phrase before—and likely word-for-word at that (or almost, anyway; just replace "hat" for any object—keys, phone, glasses, armadillo). The absence of a hat is not necessarily the most interesting part of our problem here, though it does play a crucial role in it; instead, our chief concern lies in trying to determine how long, on average, a boy who plays baseball will go out and back into his house, before he runs out of hats and yells this phrase in distress.

Picture a house with two doors; for the sake of simplicity, we can think of them as the front and back doors. Now imagine a boy, call him Pythagorous (Py for short), probably about 10 years old, who plays baseball. Every day that he practices, he wears a hat. Because of the tendency of 10-year old boys to hurry through life and quite frequently be late, Py and his mom have developed a system. They begin baseball season with the same number of hats at both the front and the back door. When Py leaves for practice, he chooses a door at random, grabs a hat from that pile, and goes on his merry way; when he returns, sweaty and exhausted at night, he cannot remember for sure which door he exited out of earlier; so he just chooses a door again at random, this time adding his hat to the pile by said door. Each day, the pattern continues.

Eventually, Py could run into the unfortunate inevitability of trying to pick up a hat off a pile that has none in it. When he does so, he will most assuredly yell that phrase, "Mom! Where's my hat!" His exclaiming thusly is a given in this scenario (it will happen); however, the problem we are concerned with is how long the boy will carry on in this fashion before exercising his lungs in the aforementioned manner.

Finally, because variety is the spice of life and we as mathematicians are always seeking ways in which a problem is affected by the slightest change, we will also consider a few variations or the problem and their results. For example, what is the expected range for the number of times he travels through his cycle before running out of hats? What if there is a third door (the side door)? n doors? What happens should the boy lose his hat between doors (as boys inevitably will)? Math can help us answer these questions; unfortunately, it will not necessarily prove helpful to the mom who must deal with the boy distraught over not finding his hat.

2 Definitions and Development

As we will soon discover, life with a ten-year old baseball player can get pretty hectic and things (hats!) can quickly be lost in the shuffle. To help us keep everything straight, we introduce some notation.

We will refer to the front door as A and to the back door as B (and any further doors in like manner and in succession: C, D, and so on and so forth); it follows then that we label the number of hats by each door with the respective door's lower case mate.

The number of hats with which we start at each door is labeled $\boldsymbol{n}$, so there will be n hats at the front door and n hats at the back door when we begin.

Definition 1 A **state, $\boldsymbol{S}$,** (also called a partition) is an arrangement of the hats by the doors. In our exploration we will only consider unique and relevant configurations, or states.

Each state illustrates the position of the hats by the doors, so when we have two doors, each state is represented as (a, b). Similarly, when we have three doors, we will have states represented as (a, b, c). For instance, when there is one hat by one door, two hats by another door, and zero hats by the third door, we have the state $(1,2,0)$.

Definition 2 A special type of state, the **absorbing state**, signifies the end of the problem. We enter the absorbing state when we try to exit a door that has zero hats by it.

When we are in the absorbing state our probability of staying in that state is 1.

Definition 3 Given state S, we define H_S as the average number of times it will take the boy to travel out of the house and back in before running out of hats.

So if $S = (a, b, c)$, then $H_{(a,b,c)}$ will be the average number of times it takes the boy to move from state S into the absorbing state. Note that in the two door case, where $m \geq k, m + k = 2n$, (m, k) can be characterized by just k, the minimum number of hats by a door. When we consider a two-door example, we use H_k and $H_{(m,k)}$ interchangeably. We represent the absorbing state by H_*.

Furthermore, we define one unit of time (one unit of H_S) by the boy going out a door and back in through a possibly different door. Another way to think about this is that we increase H_S by one each time the boy goes out a door.

In our consideration of hats, we are interested primarily in the probability of moving from one state to the next and so will need a way of picturing and organizing our results.

Definition 4 A **Markov Chains** shows each state and the probability of moving from one to the next, represented by arrows drawn from the original state to the next.

Example 5 Consider the Markov Chain for the case when $n = 1$ and we have 2 doors. We see what this looks like in **Figure 6.** Each edge is labeled with the probability of moving in the direction of the arrow from one state to the next. For example, the probability of moving from $(1,1)$ to $(2,0)$ is $\frac{1}{2}$ because there are four different ways of exiting and reentering the house, two of which will take you to the other state.

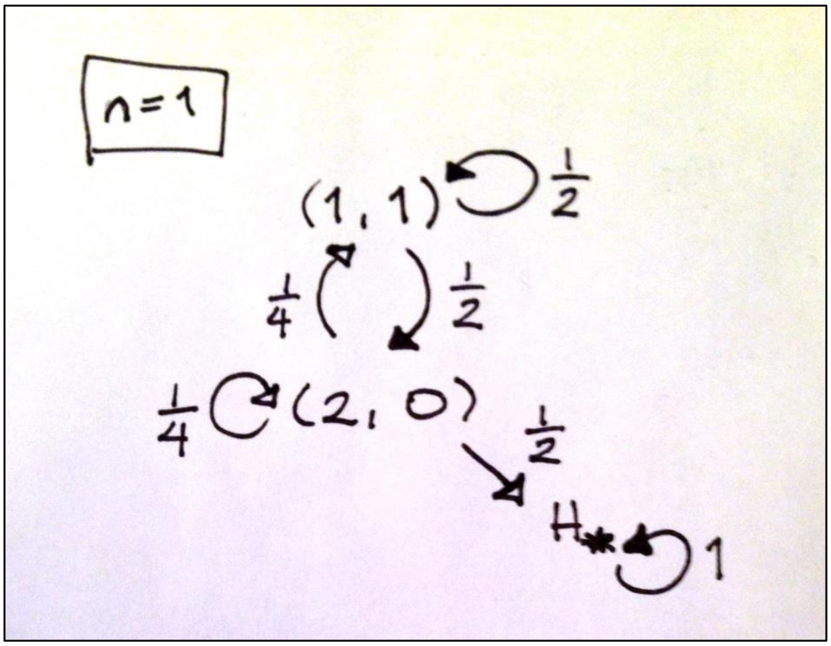


Figure 6: A Markov Chain for two doors starting with one hat by each; there are two possible states. Each color represents the probability of moving from that correspondingly colored state to a different one.

Definition 7 We translate the probabilities found in the Markov Chains into **transition matrices**. The rows represent the starting state and the fraction displayed in each position is the probability that the row heading state will move to that column's state.

Therefore, the sum of each row should equal 1. Our matrices will always be square and the last row will always be filled with zeroes except for the last entry which will be a 1, because the probability of leaving the absorbing state is 0 and the probability of staying in it is 1. Matrices allow us to more easily find the average and ranges for each case.

Example 8 Revisit our example with two doors and $n = 1$. **Figure 9** shows what the corresponding Matrix would look like.

These should be all of the definitions and terms we need in our consideration.

$$\begin{array}{c} \\ (1,1) \\ (2,0) \\ H_* \end{array}\begin{array}{c} \begin{array}{ccc} (1,1) & (2,0) & H_* \end{array} \\ \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

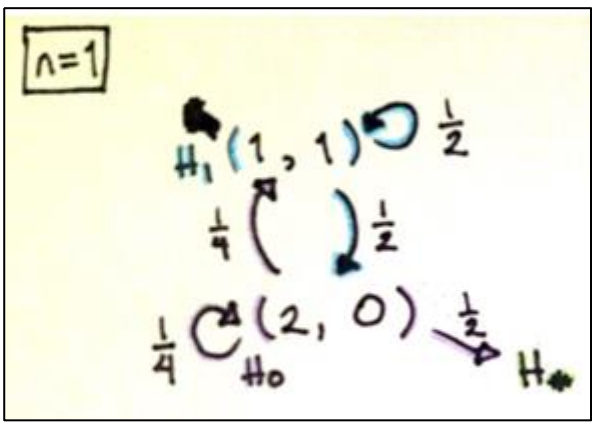
Figure 9: For two doors starting with one hat by each, the probabilities of moving between states are displayed in the matrix above.

3 Results

Now our investigation begins by looking at the base problem wherein we have two doors for the boy to choose from.

Through sheer determination and brute force (and with a little help from Markov chains), we calculate the first few cases ($n = 0,1,2,3$) by hand. Beginning with the first possible case, where we have zero hats by each door, we find that $H_0 = H_{(0,0)} = 1$. The boy has to go out the door at least once, but since he will not find a hat, he immediately enters the absorbing state, the cycle ends, and so does our problem.

More interesting to our investigation are the cases following (that is, the cases where actual hats are involved). Look at the Markov Chain and corresponding matrix when $n = 1$ in **Figure 10**.

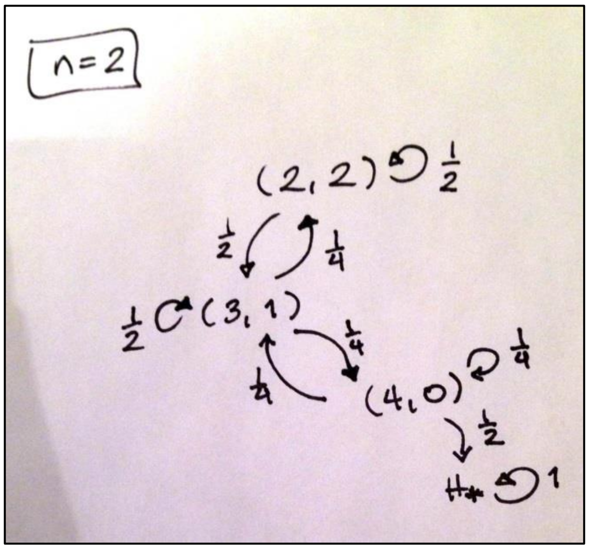


$$\begin{array}{c} \\ (1,1) \\ (2,0) \\ H_* \end{array}\begin{array}{c} \begin{array}{ccc} (1,1) & (2,0) & H_* \end{array} \\ \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

Figure 10: The Markov chain and corresponding matrix for the case with two doors and $n = 1$ hat by each door.

In the infancy of our investigation we use system of equations to solve for the average number of times, calculating an $H_{(2,0)} = H_0 = 3$ and an $H_{(1,1)} = H_1 = 5$. We can use Mathematica to solve for the average and variance of matrices. When we plug in our matrix we confirm that the average is 5 and find that the variance for one standard deviation is $\sqrt{12}$. Therefore, on average the boy will go out and back in his house 5 times before running out of hats in the stack by the door he exits; furthermore, assuming a normal distribution, 68.2% of the time our value for H_1 will fall between 1.5 and 15.5.

As we add hats we will start to see both our Markov Chains and matrices expand. Such an expansion is seen in the case of $n = 2$, in **Figure 11**.



$$\begin{array}{c} \\ (2,2) \\ (3,1) \\ (4,0) \\ H_* \end{array} \begin{array}{c} \begin{array}{cccc} (2,2) & (3,1) & (4,0) & H_* \end{array} \\ \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Figure 11: The Markov chain and corresponding matrix for the case with two doors and $n = 2$ hat by each door.

Similarly to the $n = 1$ case, we calculate using systems of equations that $H_0 = 5$, $H_1 = 11$, and $H_2 = 13$.

We again verify this using our program and calculate the range. When we start with 2 hats by both the front and back doors we find that it will take on average 13 times before the stack runs out of hats and that $68.2\ \%$ of the time H_2 will fall between 3 and 23.

We continue to sketch out the Markov Chains for the first six cases, and, noticing a pattern, conjecture a Matrix for the nth case, shown in **Figure 12**.

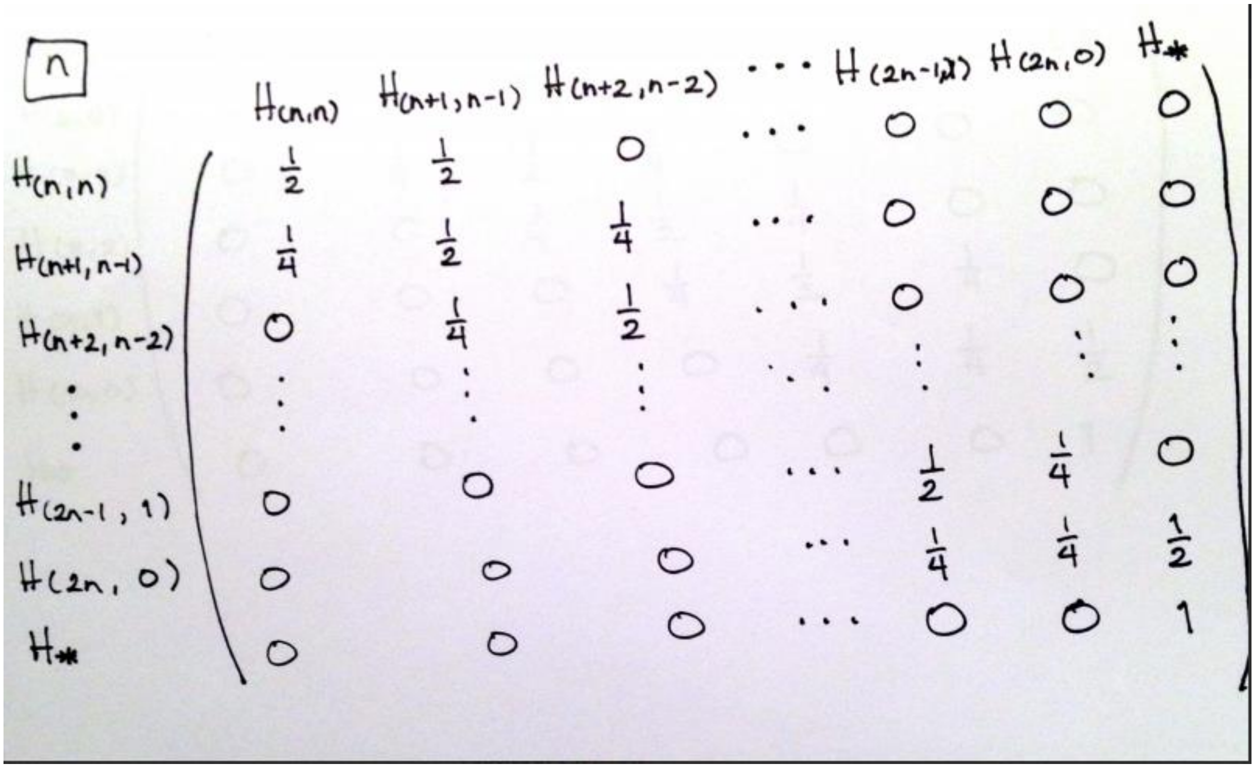


Figure 12: The proposed matrix for two doors with n hats by each door to begin.

Returning to our calculations, we use the program in conjunction with systems of equations to find the average for the first six cases. **Table 13** summarizes our findings.

n	H_n
0	1
1	5
2	13
3	25
4	41
5	61

Table 13: The average number of times, $H_{(n,n)} = H_n$, that the boy will go in and out given n hats at each door to start with.

From this matrix, we glean a few recursive relationships that aid in our calculations and lead to a closed form formula. Letting j represent the fewest number of hats by any of the doors we find: $H_0 = 1 + \frac{1}{4}H_0 + \frac{1}{4}H_1$ (1); $H_j = 1 + \frac{1}{4}H_{j-1} + \frac{1}{2}H_j + \frac{1}{4}H_{j+1}, 0 < j < n$ (2); and $H_n = 1 + \frac{1}{2}H_{n-1} + \frac{1}{2}H_{n-2}$ (3).

However, even these recursions are not as simple as we would like. Even still, they are useful in testing our conjecture and showing that formula, $H_n = 2n^2 + 2n + 1$ (4), dependent only on the number of hats that we start with, stands on its own.

This leads us nicely to our theorem.

Theorem 1 (Baseball Hats Theorem) Given two doors with n hats by each, the average number of times it will take before there are zero hats by the door the boy exits is: $H_n = 2n^2 + 2n + 1$.

Proof. We begin with a reminder of what we are trying to solve. We want to find $H_n \equiv H_{(n,n)}$, but we must careful to note that $H_{n-1} \equiv H_{(n-1,n+1)} \not\equiv H_{(n-1,n-1)}$. Thus we cannot simply use induction (though it would be nice if we could); instead, we construct our proof, using our recursive relationships gleaned from Figure 12 to do so.

Taking our recurrence relations, multiplying by 4, and rearranging, we find:

$$\begin{aligned}
4 &= 3H_0 - H_1 \\
4 &= -H_0 + 2H_1 - H_2 \\
4 &= \quad\quad - H_1 + 2H_2 - H_3 \\
&\quad\quad\quad\vdots \\
4 &= -H_{k-1} + 2H_k - H_{k+1}, for\ 2 \leq k < n \\
&\quad\quad\quad\vdots \\
2 &= \quad\quad\quad H_n - H_{n-1}.
\end{aligned}$$

Now from this set of equations, we need to establish the following equations: $H_0 = 2n + 1$ (5), $H_1 = 2 + (n-1)4 + H_0$ (6), and $H_{n-r} = H_{n-r-1} + 4r + 2$ (7).

First, note that equation 5 represents the scenario where we have come down to a state where one of the doors has only one hat by it. When we get to such a state, we assert that it will take, on average, $2n + 1$ more times to get to the absorbing state. We show this is true by looking back at out recurrence relations, cancelling everything we can, and then simplifying.

$$\begin{aligned}
4 &= 3H_0 - H_1 \\
4 &= -H_0 + 2H_1 - H_2 \\
4 &= \quad\quad - H_1 + 2H_2 - H_3 \\
&\quad\quad\quad\vdots \\
4 &= -H_{k-1} + 2H_k - H_{k+1}, for\ 2 \leq k < n \\
&\quad\quad\quad\vdots \\
2 &= \quad\quad\quad H_n - H_{n-1} \\
\hline
4n + 2 &= 2H_0 \\
H_0 &= 2n + 1.
\end{aligned}$$

Now, continuing, we use all but the first row of our recurrence relations scheme and cancel to find equation 6.

$$
\begin{aligned}
4 &= 3H_0 - H_1 \\
4 &= -H_0 + 2H_1 - H_2 \\
4 &= \quad - H_1 + 2H_2 - H_3 \\
&\vdots \\
4 &= -H_{k-1} + 2H_k - H_{k+1}, for\ 2 \leq k < n \\
&\vdots \\
2 &= \quad H_n - H_{n-1}
\end{aligned}
$$

$$
\begin{aligned}
&2 + (n-1)4 = -H_0 + H_1 \\
&H_1 = 2 + (n-1)4 + H_0.
\end{aligned}
$$

Lastly, we establish formula 7 by working backwards through our recurrence relations. We can derive this formula 7 by summing from the bottom up to any given row, and choosing the appropriate value for r, which is the difference between n and k (the current state/row we are looking at). Consider it from a more constructive angle. Starting with the last row, we add the one before it:

$$
\begin{aligned}
&2 = \ - H_{n-1} + H_n \\
&4 = -H_{n-2} + 2H_{n-1} - H_n \\
&\Downarrow \\
&4 + 2 = -H_{n-2} + H_{n-1}.
\end{aligned}
$$

Continuing,

$$
\begin{aligned}
&4 + 2 = -H_{n-2} + H_{n-1} \\
&4 = -H_{n-3} + 2H_{n-2} - H_n \\
&\Downarrow \\
&2(4) + 2 = -H_{n-3} + H_{n-2}.
\end{aligned}
$$

We begin to see that each time we are essentially cancelling new terms and adding another 4. In the rth case we would be adding $r(4)$ plus that 2 from our base recursion. Therefore, in general, we have

$$4r + 2 = H_{n-r-1} - H_{n-r}$$

$$H_{n-r} = H_{n-r-1} + 4r + 2.$$

Now that we have adequately established these three formulas, we use them, building up from the beginning, and see that:

$$H_0 = (2n + 1)$$
$$H_1 = 2 + (n - 1)4 + (2n + 1)$$
$$\vdots$$
$$H_r = 2r + \left(nr - \sum_1^r j \right) 4 + (2n + 1)$$
$$\Rightarrow H_{(n,n)} = 2n + \left(n^2 - \sum_1^n j \right) 4 + (2n + 1)$$
$$H_{(n,n)} = 2n + n^2 - \left(\frac{n^2 + n}{2} \right) + (2n + 1)$$
$$H_{(n,n)} = 2n^2 + 2n + 1,$$

as desired. Therefore, given two doors with n hats by each to begin, we know that $H_n = 2n^2 + 2n + 1$. ■

Having thus established a theorem for determining the average end to our problem with hats, we move on to consider other nuances underlying this problem and knocking at the door(s).

We have an average, but surrounding that average are a number of varying results. Sometimes the boy will go out one door and in the other repeatedly, running out of hats very quickly; other times he might get stuck in an indefinite loop going out and in the same door time and again, causing H_n to be far higher than the average. We see this play out on our screen during the simulation as it spits out an array of numbers, usually fairly consistent and close to the average, but occasionally far below or above; these outliers are where we turn our attention nest. Our question, then, is to find the variance is for each case, that is, to determine a range of values for H_n. Using a code that solves for both the average and the variance of Markov chains, we do just that and compile our findings in the table in **Table 14**.

n	$H_{(n,n)}$	**Standard Deviation**	**Approximate Range (68.2% of trials will fall between these two numbers)**
1	5	$\sqrt{12}$	1.54 – 15.46
2	13	$\sqrt{100}$	$3 - 23$
3	25	$\sqrt{392}$	$5 - 45$
4	41	$\sqrt{1080}$	$8 - 74$
5	61	$\sqrt{2420}$	$12 - 110$
6	85	$\sqrt{4732}$	$16 - 114$

Table 14: A table detailing the average, standard deviation, and approximate range for the first 6 cases of the two door hat problem.

Zooming in on the case where $n = 2$, let us see more closely how the values of H_n vary over the course of 100 trials in **Figure 15**. The box outlines our calculated range, and we see that the majority of trials fell in that range as expected.

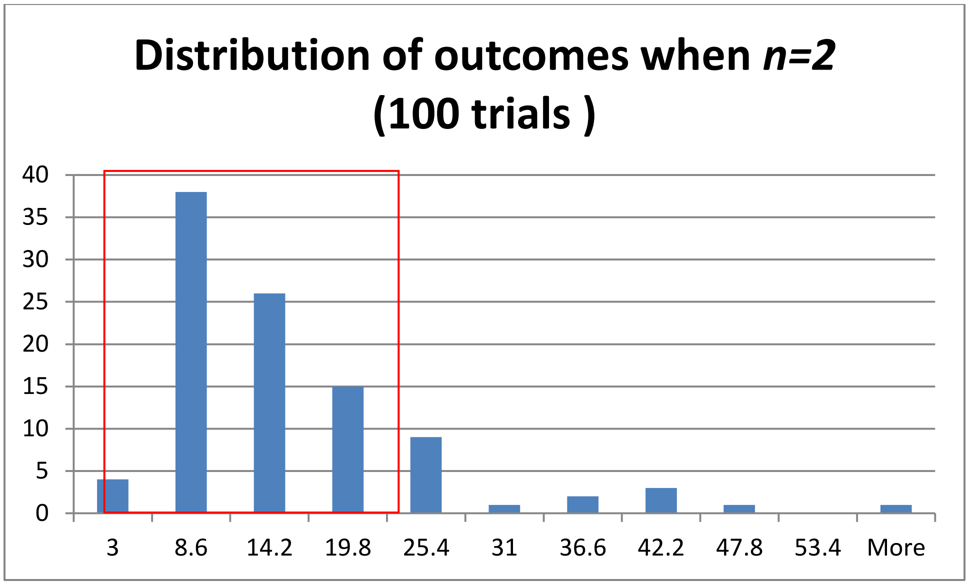


Figure 15: A histogram displaying the frequency of the different values of H_n over 100 trials.

After we have figured out a formula and calculated averages and variances for our problem with two doors, Py's family decides to do some home improvement and add a side door for better ease of access to the driveway. How will the addition of a third door affect our problem?

With an added possible location for our hats to be stacked, imagining this scenario becomes just a touch more difficult. To help, we return to our definition of states, and recognize that we only need to focus on the relevant or unique arrangements. We begin, then, by figuring out how many different partitions, or states, we can divide $3n$ hats between. **Table 16** lists the number of relevant partitions for the first several cases.

As you can probably guess, our Markov chains will quickly become very complicated and intricate to sketch out by hand and it is unclear whether a pattern will be so easily deduced as in the problem with only two doors. We present the first two Markov chains and their translated Matrices for consideration now in **Figures 17** and **18**.

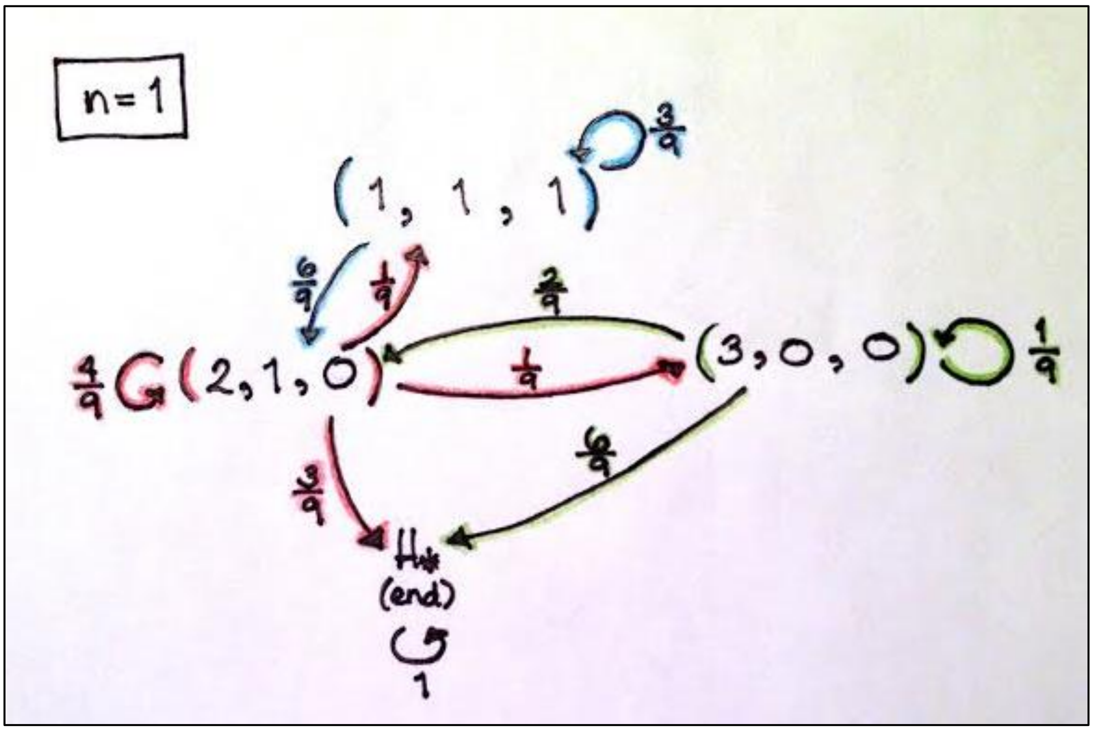
Because we have these first two cases in matrix form, we can use our program to solve for the average and variance. Our results are listed in **Table 19**.

We use our C++ simulation code to generate more data for the three door case, but that is about as far as our present examination will take us.

We now turn our attention towards picking up our hats, making sure they are all in a row, and considering various areas for future research.

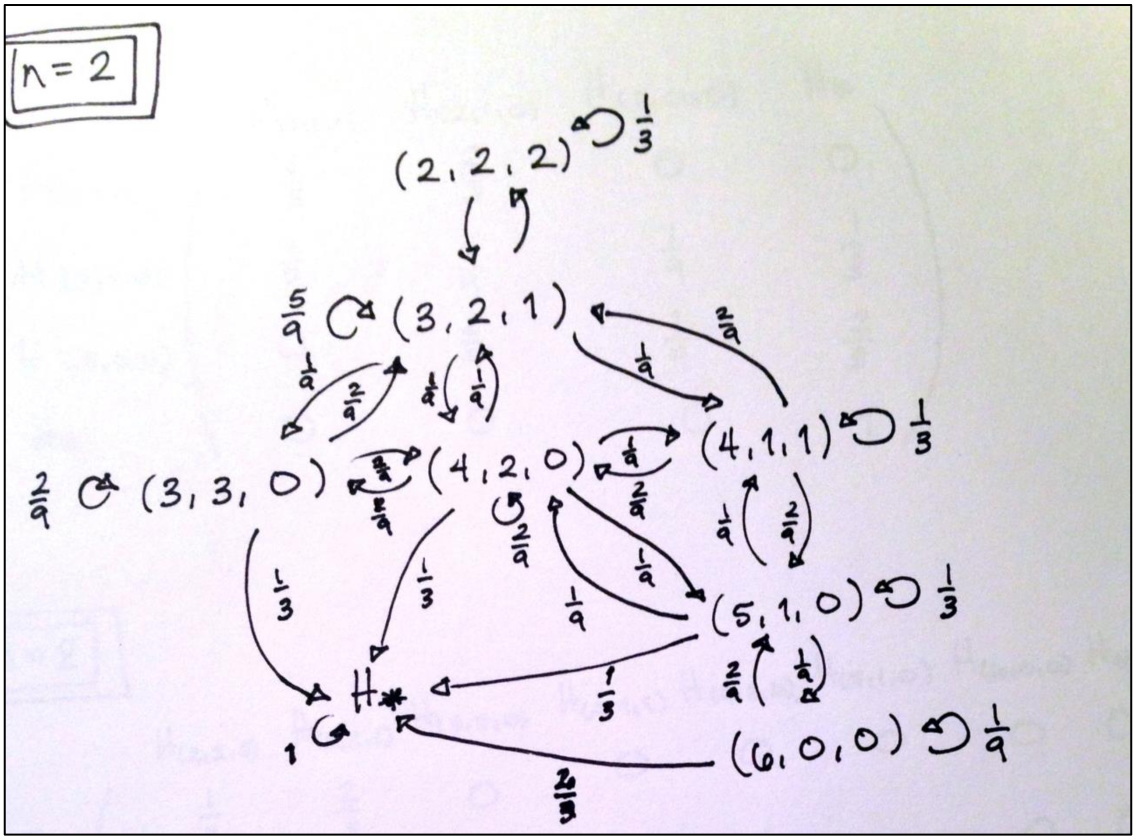
n	partitions
0	1
1	3
2	7
3	12
4	18
5	27

Table 16: The number of relevant states (partitions) for the first 6 cases with three doors. These are the number of different ways we can arrange n hats amongst three doors.



$$\begin{array}{c} \\ (\mathbf{1},\mathbf{1},\mathbf{1}) \\ (\mathbf{2},\mathbf{1},\mathbf{0}) \\ (\mathbf{4},\mathbf{3},\mathbf{0},) \\ \boldsymbol{H}_* \end{array} \begin{array}{c} \begin{array}{cccc} (\mathbf{1},\mathbf{1},\mathbf{1}) & (\mathbf{2},\mathbf{1},\mathbf{0}) & (\mathbf{3},\mathbf{0},\mathbf{0}) & \boldsymbol{H}_* \end{array} \\ \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{9} & \frac{4}{9} & \frac{1}{9} & \frac{1}{3} \\ 0 & \frac{2}{9} & \frac{1}{9} & \frac{2}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Figure 17: The Markov chain and corresponding Matrix for three doors starting with one hat by each door.



$$
\begin{array}{c|cccccccc}
 & (2,2,2) & (3,2,1) & (3,3,0) & (4,1,1) & (4,2,0) & (5,1,0) & (6,0,0) & H_* \\
\hline
(2,2,2) & \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
(3,2,1) & \frac{1}{9} & \frac{5}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & 0 & 0 & 0 \\
(3,3,0) & 0 & \frac{2}{9} & \frac{2}{9} & 0 & \frac{2}{9} & 0 & 0 & \frac{1}{3} \\
(4,1,1) & 0 & \frac{2}{9} & 0 & \frac{1}{3} & \frac{2}{9} & \frac{2}{9} & 0 & 0 \\
(4,2,0) & 0 & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{2}{9} & \frac{1}{9} & 0 & \frac{1}{3} \\
(5,1,0) & 0 & 0 & 0 & \frac{1}{9} & \frac{1}{9} & \frac{1}{3} & \frac{1}{9} & \frac{1}{3} \\
(6,0,0) & 0 & 0 & 0 & 0 & 0 & \frac{2}{9} & \frac{1}{9} & \frac{2}{3} \\
H_* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

Figure 18: The Markov chain and corresponding Matrix for three doors starting with one hat by each door.

n	$H_{(n,n,n)}$	Standard Deviation	Approximate Range (68.2% of trials will fall between these two numbers)
1	4.6	$\sqrt{7.74}$	$1.82 - 7.38$
2	11.125	$\sqrt{54.4219}$	$3.75 - 18.50$
3	20.7248	$\sqrt{202.261}$	$6.50 - 41.45$

Table 19: A table detailing the average, standard deviation, and approximate range for the first 3 cases of the three door hat problem.

n	Hn		n	Hn		n	Hn		n	Hn
0	1		8	113.2		16	416.93		24	913.65
1	4.6		9	140.75		17	468.97		25	989.34
2	11.13		10	171.14		18	523.48		26	1067.24
3	20.65		11	204.49		19	580.82		27	1149.25
4	33.15		12	241.21		20	641.25		28	1234.88
5	48.66		13	280.55		21	705.37		29	1320.32
6	67.26		14	323.12		22	770.46		30	1411.26
7	88.62		15	368.58		23	840.39			

Table 20: The average number of times, $H_{(n,n,n)}$, that the boy will go in and out given n hats to start with at each of the three doors.

4 Conclusion and Directions for Further Research

Thus, we have determined that the problem with hats lies in their tendency to run out (at least under the current constraints of our problem). In our exploration we looked at what happens when we start out with the same number of baseball hats stacked by each door, using Markov chains and matrices to map out the number of times it takes the young baseball player to exit and reenter his house before running out of hats by the door he exits. From these representations, we were able to calculate the expected averages and ranges for each value of n hats. Noticing a pattern and determining recurrence relationships, we used induction to prove that the average number of times, the value of H_n, can be calculated using the formula $H_n = 2n^2 + 2n + 1$. We wrapped up our present perusal by calculating the first several values of $H_{(n,n,n)}$ in the three door case; a full and complete examination of this case similar to the one we launched with the two door case, however, is beyond the scope of the present project.

Indeed, there are several considerations that lay beyond that scope. Before we put down our calculators/pens, we turn our gaze forward and look at these directions for further research. These areas of further research include, but are not limited, to the following: adding doors (a more in depth look at three doors and then a similar examination of four, five, or even m doors); introducing the very likely probability that the boy could lose a hat (or come back with extra hats!); keeping the number of hats we start out with by each door consistent at just $n = 1$; but increasing the number of doors; and starting out the season (problem) with an uneven distribution of hats by the doors of the house.

First, a more in depth look at the three door case would require more careful detailing of Markov chains and matrices. From matrices (and their calculated expected averages), perhaps we could find a pattern that might lead to a recursive relationship, and further to a closed-form formula that would determine the average for each value of n hats that we start out with by each door. Additionally, we could continue to calculate the range as we have begun to do.

Furthermore, we could extend this variation indefinitely, looking at what happens with four, five, or even m doors. Such an exploration would undoubtedly become increasingly complicated and difficult to organize, as we have seen with the difficulty just jumping from two doors to three.

Next, we might look at what happens when Py goes to practice and comes back without the hat he brought—or comes back with extra hats. These real life considerations would help add validity to our problem. We would draw out the Markov chains in a similar fashion, this time introducing the chance that at each state Py could lose (or gain) a hat with probability p. This necessarily increases the number of states and therefore also the complexity.

Additionally, we might consider what happens when we start out with one hat by each door. We never increase the number of hats; we do, however, increase the number of doors. Again, a similar use of Markov Chains and matrices helps to illustrate this problem and support the calculations the average and range for each case.

We could also look at what happens when we start with an uneven state, that is when each door does not start with the same number of hats. Essentially this variation is similar to just starting in the middle of our present problem. Conceivably, we could solve this problem with systems of equations, at least in the first several cases.

Most interestingly, we should look at why an induction proof of our theorem also yields the same results, even though it uses a slightly different meaning of H_n. That is, our original proof very nicely used induction to show $H_n = 2n^2 + 2n + 2$; however, that induction proof mistakenly used the building blocks, $H_{n-i} \equiv H_{(n-i,n-i)}$, when that is simply not the case; it should be, rather, that $H_{(n-i,n+i)}$, because at any given moment in our problem we will always have n hats. Is it merely a strange coincidence that both nuances of the proof work? Or is there something more underneath? Indeed, this mystery surrounding the proof is one worth solving and might be where we turn our attention to next, should we proceed with our present consideration.

Finally, we might try to code the simulation in a program with graphics to visualize these variations in real time.

Hence, we have reached the end of our problem for now, with a core solution to the case with two doors and n hats—though perhaps not quite the solution Py and his mom are looking for.

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