Factoring 2x2 Matrices with Determinant of ± 1

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Abstract

We examine properties of multiplying and factoring 2×2 matrices to prove that there exist two generators to which all other 2×2 matrices factor. We prove that all matrices fitting out specifications must have a dominant column and can be factored uniquely to a product of our two generators.

1 Introduction

In this study we will show a factorization pattern for matrices. More specifically, we will prove a factorization pattern for 2x2 matrices with non-negative integer entries and a determinant of positive or negative one. We will prove multiple theorems which are used to show that all matrices fitting our guidelines can be factored down to two unique matrices.

For the basis of this study, let A be a 2x2 matrix with integer entries and determinant ± 1. To start, we will assume that all of the entries in A are non-negative. We will show that A can be written uniquely as a product of the matrices $y = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ up to the fact that $x^2 = I$, the identity matrix.

2 Definitions and Development

We begin with important definitions and examples.

Definition 1 A **matrix** is a rectangular array of numbers. The numbers in the array are called the **entries** in the matrix.

Example 2 The array $y = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ is a matrix. The numbers 1, 1, 1, and 0 are the entries of the matrix.

Definition 3 Let A be a square matrix. The **determinant function** is denoted by $\det$, and we define $\det(A)$ to be the sum of all signed elementary products from A. The number $\det(A)$ is called the **determinant of A**.

Lemma 4 (Determinant of 2x2 Matrix) If A is a 2x2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the determinant of A, denoted $\det(A)$, is $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - cb$.

Definition 5 If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the **product** AB is the $m \times n$ matrix whose entries are determined as follows. To find the entry in row i and column j of AB, single out row i from the matrix A and column j from the matrix B. Multiply the corresponding entries from the row and column together and then add up the resulting products.

Example 6 The product of $y = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Definition 7 The **identity matrix**, denoted I, is a square matrix with 1's on the main diagonal and 0's off the main diagonal. The effect of multiplying a given matrix by an identity matrix is to leave the given matrix unchanged.

Example 8 The 2x2 identity matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Definition 9 If A is a square matrix, and if a matrix B can be found such that $AB = BA = I$, then B is called an **inverse** of A.

Example 10 Given $y = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ is a square matrix and $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$, thus we see that $y^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$.

Definition 11 A matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has a **dominant column** if $a \geq b$ and $c \geq d$ or $b \geq a$ and $d \geq c$.

Example 12 The matrix $A = \begin{bmatrix} \mathbf{5} & 3 \\ \mathbf{2} & 1 \end{bmatrix}$ has a dominant column because $5 \geq 3$ and $2 \geq 1$.

Note that $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$, and $\det \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix} = (5 \cdot 1) - (3 \cdot 2) = 5 - 6 = -1$.

Example 13 The matrix $B = \begin{bmatrix} 5 & 6 \\ 3 & 2 \end{bmatrix}$ does not have a dominant column.

Notice that $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} - ad - bc$, and that $\det \begin{bmatrix} 5 & 6 \\ 3 & 2 \end{bmatrix} = (5 \cdot 2) - (6 \cdot 3) = 10 - 18 = -8$.

3 Results

Now we present the main results of this study.

Theorem 14 (Dominant Column Theorem) Every 2x2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with non-negative integer entries and a determinant of ± 1 must have a dominant column, except the identity matrix and $x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. That is, either $a \geq b$ and $c \geq d$ or $b \geq a$ and $d \geq c$.

Proof. Suppose that $a > b$ and $d > c$, which implies that $ad > bc$. It is also true that $a \geq b + 1$ and $d \geq c + 1$. Note that $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \pm 1$. We know that $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc > 0$ which implies that $ad - bc = 1$. This implies that $ad = 1 + bc$. Since $a, d = 1$ and $b, c = 0$ generates the identity matrix, and $a, d = 0$ and $b, c = 1$ generates x, we will assume that either $a \geq 1$ or $d \geq 1$ and either $b \geq 1$ or $c \geq 1$. Then, $ad \geq (b + 1)(c + 1)$ which implies that $ad \geq bc + b + c + 1$. This implies that $ad - bc \geq b + c + 1 > 2$ thus implying that $ad - bc > 2$, a contradiction. Not having a dominant column results in a determinant not equal to ± 1. Therefore, there must be a dominant column. ∎

Theorem 15 (Factorization Theorem) Every 2x2 matrix with non-negative integer entries and a determinant ± 1, except for the identity matrix, can be factored uniquely into a product of $x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $y = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

Proof. We know that we must have a dominant column by the Dominant Column Theorem. Note that all multiplication will be applied on the right. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If the left column of A is dominant, then $A = By$ where $B = \begin{bmatrix} b & a-b \\ d & c-d \end{bmatrix} \geq 0$. If the right column of A is dominant, then $A = Bx$ where $B = 0$ and the left column of B is dominant. Repeat this process on B and continue until it ends with either x or y. Note that only 5 matrices have $\det(A) = \pm 1$ and all entries either 0 or 1; namely $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathrm{I}$, $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = xy$, and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = yx$. The process will end with either x or y. Each matrix can be factored uniquely because there is only one action at each step of the process. If the left column of A is dominant, we multiply B by y. If the right column of A is dominant, we multiply B by x. We cannot have two dominant columns, so the proof is complete. ■

Example 16 We will factor the matrix $J = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$.

The matrix J has a dominant right column, therefore we multiply by x. The product matrix has a dominant left column and therefore we multiply by y^{-1}. The product matrix of that has a dominant left column, thus we multiply by y^{-1} again. The product matrix again has a dominant left column, and so we multiply by y^{-1}. The next product matrix has a dominant right column, therefore we multiply by x. The final product is our original matrix y.

$$\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

This shows that $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}$, and $\begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$.

Therefore, $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, or $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = yxyyyx$.

4 Conclusion and Directions for Further Research

We have proven that there exist two generators to which all 2x2 matrices factor. In the future, we can examine the different uses for this factorization. We can explore how to use factorization in proving similarity. Specifically, we can research if it is possible to determine if two matrices are similar over integers. Also, in this study, we only researched matrices with non-negative entries. If the future, we could study the factorization of matrices with negative entries.

References

[1] Anton, H., *Elementary Linear Algebra 5e*, John Wiley & Sons, New York, 1987.