Rolling Cups and Geometry

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Abstract

We analyze the circular trajectory of a cup placed on its side and rolled. Using measurable components of the cup such as the radii and height we find the equation of the circle traced by the cup.

1 Introduction

Cups have many different purposes, from a tool to drink out of, to a source of entertainment, to a musical instrument. Another purpose for a cup involves math, specifically geometry. Have you ever wondered what happens when you place a cup on its side and tap it? It rolls in a circular trajectory. The size of this circle depends on three measurable components of the cup: the height, the radius of the top of the cup, and the radius of the bottom of the cup. Our goal in this paper is to determine the radius of the path of the cup, and thus the equation for the circle it creates using the radii of the cup and the height.

2 Definitions and Development

We will be using a frustum of a cone as the shape for the cup in this proof, as shown in Figure 1. If the cup is not a frustum of a cone, using the outermost points when place on its side we will create a frustum of a cone, similar to that in Figure 1.

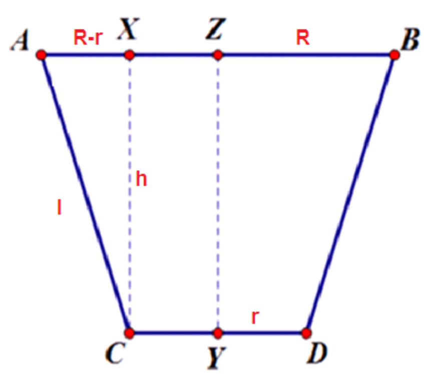

Figure 1 A two-dimensional version of the cup

We now define critical areas of the cup that will be used throughout this paper.

Definition 1 A **large radius** (or **upper radius**), denoted R, is the distance from the center of the top of the cup to the outer edge of the top of the cup. In Figure 1, this radius is represented by the lengths of segments $\overline{AZ}$ and $\overline{ZB}$.

Definition 2 A **small radius** (or **lower radius**), denoted r, is the distance from the center of the bottom of the cup to the outer edge of the bottom of the cup. In Figure 1, the small radii are the lengths of segments $\overline{CY}$ and $\overline{YD}$.

Definition 3 The **slant height**, denoted l, is the distance from the upper edge of the cup to the lower edge of the cup. In Figure 1, this slant height are the lengths of segments $\overline{AC}$ and $\overline{BD}$.

Definition 4 The **height** of a cup, denoted h, is the distance from the center of the top of the cup to the center of the bottom of the cup. In Figure 1, it is the length of segment $\overline{ZY}$.

We will need the following theorem for our proof.

Theorem 5 (Pythagorean Theorem) In a right triangle, the area of the square whose side is the hypotenuse is equal to the sum of the areas of the squares whose sides are the two legs.

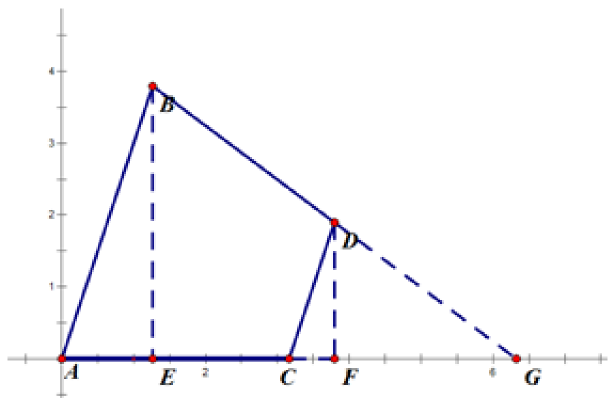

Figure 2 A two-dimensional cup placed at the origin.

The next two definitions are primary topics in this research.

Definition 6 The **center of the circle** will be point G from Figure 2 and is the center of the circular trajectory a cup makes.

Definition 7 The **circle radius** is the radius from the out most point on the cup to the center of the circle. In Figure 2, it is the length of segment $\overline{AG}$.

3 Results

We now present our main theorem.

Theorem 8 If the height, the large radius, and the small radius of a cup are h, R and r respectively, the circle radius will equal

$$\frac{R \cdot \sqrt{h^2 - (R-r)^2}}{R-r}.$$

This can also be written in terms of the large radius, small radius and slant height, R, r and l respectively as

$$\frac{R \cdot l}{R - r}.$$

Proof. Let diagram $ABCD$ represent a cup where $\overline{AB}$ is parallel to $\overline{CD}$ as shown in Figure 1. Let the large radii $\overline{AZ}$ and $\overline{ZB}$ have a length R and the small radii, $\overline{CY}$ and $\overline{YD}$ have a length of r. Let $\overline{ZY}$ be perpendicular to $\overline{CD}$ and $\overline{AB}$ and have a length h, the height of the cup.

Let $\overline{XC}$ be perpendicular to $\overline{CD}$ with a length h. We know $\overline{XC} = h$ and $\overline{AX} = R - r$. We can the find $\overline{AC}$, the slant height of the cup, using the Pythagorean Theorem. We call the slant height l and thus

$$l = \sqrt{h^2 + (R - r)^2}.$$

Using this length, and trigonometric properties, we can find the measure of $\angle BAC$ such that

$$m\angle BAC = \sin^{-1}\left(\frac{h}{l}\right) = \cos^{-1}\left(\frac{R - r}{l}\right).$$

Assume the cup is placed at the origin with point A placed at $(0,0)$ as shown in Figure 2. We know that point C has the coordinate $\left(\sqrt{h^2 + (R - r)^2}, 0\right)$ by the previous computation of l.

Let $\overline{BE}$ be perpendicular to $\overline{AC}$. We compute

$$\overline{AE} = 2R \cdot \cos(\angle BAC) = 2R \cdot \left(\frac{R - r}{l}\right)$$

and

$$\overline{BE} = 2R \cdot \sin(\angle BAC) = 2R \cdot \left(\frac{h}{l}\right).$$

Thus, point B has the coordinates $\left(2R \cdot \left(\frac{R-r}{l}\right), 2R \cdot \left(\frac{h}{l}\right)\right)$.

Let $\overline{DF}$ be perpendicular to $\overleftrightarrow{AC}$. We know $m\angle BAC = m\angle DCF$ by corresponding angles. We compute

$$\overline{CF} = 2r \cdot \cos(\angle DCF) = 2r \cdot \left(\frac{R - r}{l}\right)$$

and

$$\overline{FD} = 2r \cdot \sin(\angle BDCF) = 2r \cdot \left(\frac{h}{l}\right)$$

by previous computation and trigonometric properties. Thus, point D has the coordinates

$\left(l + 2r \cdot \left(\frac{R-r}{l}\right), 2r \cdot \left(\frac{h}{l}\right)\right)$.

We compute the slope of $\overline{BD}$ to be

$$m = \frac{2 \cdot \left(\frac{h}{l}\right) \cdot (R - r)}{2 \cdot \left(\frac{R - r}{l}\right) \cdot (R - r) - l} = \frac{2 \cdot h \cdot (R - r)}{2 \cdot (R - r)^2 - l^2}$$

To find the equation of $\overline{BD}$, we use this slope, the point B and the point-slope form to acquire the equation

$$y - 2R \cdot \left(\frac{h}{l}\right) = \left(\frac{2 \cdot h \cdot (R - r)}{2 \cdot (R - r)^2 - l^2}\right)\left(x - 2R \cdot \left(\frac{R - r}{l}\right)\right).$$

Finally, using the equation for $\overline{BD}$, we find the x-intercept. This point, G will be the center of the circle. We compute

$$0 - 2R \cdot \left(\frac{h}{l}\right) = \left(\frac{2 \cdot h \cdot (R - r)}{2 \cdot (R - r)^2 - l^2}\right)\left(x - 2R \cdot \left(\frac{R - r}{l}\right)\right) \Rightarrow$$

$$0 = \left(\frac{2 \cdot h \cdot (R - r)}{2 \cdot (R - r)^2 - l^2}\right)x - \frac{2R \cdot h \cdot l}{2 \cdot (R - r)^2 - l^2} \Rightarrow$$

$$x = -\left(\frac{-2R \cdot h \cdot l}{2 \cdot (R - r)^2 - l^2}\right) \Big/ \left(\frac{2R \cdot h \cdot l}{2 \cdot (R - r)^2 - l^2}\right) \Rightarrow$$

$$x = \frac{R \cdot \sqrt{h^2 - (R - r)^2}}{R - r} = \frac{R \cdot l}{R - r}$$

Therefore, the center of the circle will always be at point G whose coordinates are $\left(\frac{R \cdot l}{R - r}, 0\right)$ making the circle radius $\frac{R \cdot l}{R - r}$. ■

4 Conclusion and Directions for Further Research

We have proven that if the height, the large radius, and the small radius of a cup are given, the circle radius will equal the large radius multiplied by the slant height divided by the difference of the large radius and small radius. With only three measureable components, the radii and the height or slant height, we can determine the equation of the trajectory of a cup. We have created yet another function for cups, mathematical applications. This also opens the door of applying mathematical concepts to real-world examples.

An area of further research for the “cup problem” would be analyzing the equation of a circle made by a cup where the top half does not lie directly over the bottom half. This would create two different slant heights opposed to the equal slant heights we used in our cups.

References

[1] Bonola, R., *Non-Euclidean Geometry*, Dover Publications, Inc., New York, 1955.