Fundamental Groups of Simplicial Complexes

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Abstract

We define two different simplicial complexes, the common divisor simplicial complex and the prime divisor simplicial complex, from a set of integers, and explore their similarities. We will show that if one is connected, then the other is connected. We will also show that for any given set of integers, the fundamental groups of the resulting simplicial complexes are isomorphic.
KEYWORDS: Abstract Algebra, Topology, Algebraic Topology, Fundamental Group, Simplicial Complex, simplex

1 Introduction

Suppose G is a finite group. Historically, much has been deduced about the structure of G when only given information about its irreducible characters. Notationally, we write $\mathrm{Irr}(G)$ for the set of irreducible characters of a group G and we write $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$. A shocking amount of information can be deduced about the group G when only given the set $\mathrm{cd}(G)$. For a comprehensive overview, see [3].

Two useful ways of visualizing the members of the set $\mathrm{cd}(G)$ have frequently been employed in this area of research. The first is called the *character degree graph* of $\mathrm{cd}(G)$, denoted by $\Gamma(G)$. The vertices of this graph are the members of the set $\mathrm{cd}(G) \setminus \{1\}$, and there is an edge connecting two vertices if the corresponding irreducible character degrees have a nontrivial common divisor. The second is the *prime vertex graph*, denoted $\Delta(G)$, which has the primes dividing some member of $\mathrm{cd}(G)$ as its vertices, and there is an edge between two vertices if there is a member of $\mathrm{cd}(G)$ divisible by the two associated primes. More recent research questions have involved what can be said about the structure of G given only either $\Gamma(G)$ or $\Delta(G)$; [3] also contains a summary of these types of results.

It has long been understood that there is an intimate relationship between the graphs $\Gamma(G)$ and $\Delta(G)$. For example, it is known ([3]) that $\Gamma(G)$ is connected if and only if $\Delta(G)$ is connected, and that the distance between two vertices in $\Gamma(G)$ is closely related to the distance between "corresponding" vertices in $\Delta(G)$. The relationship between $\Gamma(G)$ and $\Delta(G)$ has given researchers the fluidity to obtain results pertaining to only one of these graphs and use the proper correspondence to obtain a result about the other.

Still more current research [2] suggests moving away from studying only the structure of the common divisor graph of G and the prime divisor graph of G toward the study of the *common divisor simplicial complex* of G, denoted $\mathcal{G}(G)$, and the *prime vertex simplicial complex* of G, henceforth $\mathcal{D}(G)$. In [2], the author works primarily with the simplicial complex $\mathcal{G}(G)$ obtaining results about the fundamental group. It is unclear in this work if there is any analogous result regarding the fundamental group of $\mathcal{D}(G)$ as no more general correspondence between $\mathcal{G}(G)$ and $\mathcal{D}(G)$ is known.

Although the connection between $\Gamma(G)$ and $\Delta(G)$ is noticeable and documented, the nature of the relationship between the two graphs is imprecise. Moreover, as work with $\mathcal{G}(G)$ and $\mathcal{D}(G)$ is just emerging, there is no current research justifying a more general relationship between these two structures. In this paper, we establish the precise nature of the relationship between $\mathcal{G}(G)$ and $\mathcal{D}(G)$ by creating a specific map η between the two simplicial complexes. This map, when restricted to $\Gamma(G)$ and $\Delta(G)$, makes precise the relationship so frequently used in the works summarized in [3]. We then proceed to establish that if G is a finite group with $\mathcal{G}(G)$ connected, then the map η induces a map η_* on the fundamental groups of $\mathcal{G}(G)$ and $\mathcal{D}(G)$, denoted $\pi_1(\mathcal{G}(G))$ and $\pi_1(\mathcal{D}(G))$, respectively. Our main result is the following.

Theorem 1. *Suppose G is a finite group with $\mathcal{G}(G)$ connected. The induced map η_* from $\pi_1(\mathcal{G}(G))$ to $\pi_1(\mathcal{D}(G))$ is a group isomorphism.*

This correspondence allows us to derive analogous results to those found in [2] for $\mathcal{D}(G)$, and it opens the door for further analogies to be drawn regarding all works involving $\pi_1(\mathcal{G}(G))$ or $\pi_1(\mathcal{D}(G))$. Having the more general map η between $\mathcal{G}(G)$ and $\mathcal{D}(G)$ allows us the ability to attempt to translate all kinds of results established on either $\mathcal{G}(G)$ or $\mathcal{D}(G)$ to the other simplicial complex.

For the majority of the paper, results are stated more generally for the common divisor simplicial complex and prime divisor simplicial complex of a set of positive integers X. In section 2, we provide the necessary topological definitions for our work. In section 3, we define the map η, establishing a correspondence between $\mathcal{G}(G)$ and $\mathcal{D}(G)$. In section 4, we show that η induces a map η_* on edge-paths in $\mathcal{G}(G)$ and $\mathcal{D}(G)$, and show that this induced map is an isomorphism between $\pi_1(\mathcal{G}(G))$ and $\pi_1(\mathcal{D}(G))$.

2 Preliminary Definitions

Definition 2. A **simplex** is the generalization of a triangle to arbitrary dimensions. An n-simplex is the convex hull of $n+1$ vertices in $\mathbb{R}^n$.

A 0-simplex is a point, a 1-simplex is a line, a 2-simplex is a triangle, and a 3-simplex is a tetrahedron. Simplices can also exist in more than three dimensions, where they take the shape of the convex hull of their vertices. All simplices are simply-connected. An n-simplex can equivalently be described as a simplex of dimension n. Any n-simplex, because it has $n+1$ vertices, can be embedded in $\mathbb{R}^{n+1}$ by assigning each vertex an axis and giving it a coordinate of 1 on that axis and 0 on every other axis.

Definition 3. A **face** of a simplex is any subset of the vertices of that simplex.

Definition 4. A **simplicial complex** is a topological space consisting of a set $\mathcal{K}$ of simplices with the following properties:

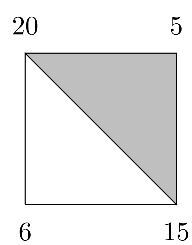
- Any face of a simplex from $\mathcal{K}$ is also in $\mathcal{K}$;
- The intersection of any two simplices in $\mathcal{K}$ is a face of both simplices. [6]

A simplicial complex can be embedded in $\mathbb{R}^n$ where n is the number of vertices in the simplicial complex. In this way we define the fundamental group, using the standard topology on $\mathbb{R}^n$. The definition and properties of the fundamental group can be found in [4].

Now let X be a set of positive integers, and let $X^* = X \setminus \{1\}$. We will define two simplicial complexes from X^*.

Definition 5. The **common divisor simplicial complex** of X, which we will denote $\mathcal{G}(X)$, has as its vertices the elements of X^*. Given a set of $n+1$ vertices $V \subseteq X^*$, we form an n-simplex out of V if $\gcd(V) > 1$.

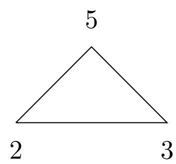
What this definition means is that we will form a simplex out of any subset of X sharing a common divisor. For example, given the set $X = \{5, 6, 15, 20\}$, $\mathcal{G}(X)$ is the following simplicial complex.



In this graph, the vertices are simplices, the edges are simplices, and the shaded region (the triangle) is a simplex. But the unshaded triangle is not a simplex because the vertices 6, 15, and 20 do not share a common divisor.

Definition 6. The **prime divisor simplicial complex** of X, which we will denote $\mathcal{D}(X)$, has as its vertices the set of primes dividing at least one member of X^*. If P is the set of primes dividing some member of X^*, and $Q \subseteq P$ consisting of $n+1$ primes, we form an n-simplex out of Q if there exists an element v of X such that every element of Q divides v.

For the same set X, $\mathcal{D}(X)$ is the following simplicial complex.



Because every vertex in $\mathcal{G}(X)$ has a corresponding integer from X, we will use the phrase "divides a vertex" to mean "divides the integer corresponding to the vertex."

Looking at examples of both simplicial complexes, we see that there is a similarity of structure between $\mathcal{G}(X)$ and $\mathcal{D}(X)$. The two are not the same, but for any given set they always seem to have the same number of loops. We will show that the two simplicial complexes have isomorphic fundamental groups. To do this, we will define a map from $\mathcal{G}(X)$ to $\mathcal{D}(X)$ that preserves the fundamental group.

3 A Correspondence between $\mathcal{G}(X)$ and $\mathcal{D}(X)$

First we define a function that we will use in our map. Let v be an element of X. By the Fundamental Theorem of Arithmetic v has a unique prime factorization. Define $\pi(v)$ to be the set of distinct primes p such that p divides v, and note that $|\pi(v)|$ is the number of distinct primes in the prime factorization of v, not including multiplicity. This means that if $p^2|v$, this term only contributes 1 to $|\pi(v)|$. For example, let $v := 2^2 \cdot 5$. Then $|\pi(v)| = 2$, one for the two 2 factors and one for the 5 factor. Given a set σ of integers, we define $\pi(\sigma)$ to be $\pi(\gcd(\sigma))$.

We are ready to define our map, which we will denote by η.

Definition 7. Given X a set of integers, let the domain of η be the set of simplices in $\mathcal{G}(X)$, let the range be the set of simplices in $\mathcal{D}(X)$, and let $\sigma = \{v_1, v_2, \ldots, v_{n+1}\}$ be an n-simplex in $\mathcal{G}(X)$. Define $\eta(\sigma)$ to be the simplex in $\mathcal{D}(X)$ with the members of $\pi(\sigma)$ as vertices.

Note that this simplex will have dimension $|\pi(\sigma)| - 1$.

The map η is not an injection for all sets. Let $X := \{2, 4, 8\}$. Then $\mathcal{D}(X)$ consists of a single point, 2, onto which all three vertices, all three 1-simplices, and the 2-simplex of $\mathcal{G}(X)$ are mapped. It is also possible to have a simplex in $\mathcal{G}(X)$ map onto a simplex in $\mathcal{D}(X)$, and another simplex in $\mathcal{G}(X)$ map onto a face of the same simplex in $\mathcal{D}(X)$.

The map η is not a surjection for all sets, but it does have some nice properties similar to a surjection. Let p be a vertex in $\mathcal{D}(X)$. Then there exists $v \in X$ such that $p|v$. We see that $\eta(v)$ is the $(|\pi(v)| - 1)$-simplex with p as one of its vertices in $\mathcal{D}(X)$. This means that not every vertex is the image of a simplex, but every vertex is a face of the image of a simplex. Likewise let σ be any n-simplex in $\mathcal{D}(X)$. This means that there exists $v \in X$ and $n+1$ primes $p_1, p_2, \ldots, p_{n+1}$ such that $p_1 p_2 \ldots p_{n+1} | v$, from which we conclude that $\eta(v)$ is either σ or a simplex with σ as one of its faces. We see that for all primes $p \in P$ there exists some simplex $\sigma \in \mathcal{D}(X)$ with p a vertex of σ such that $\sigma = \eta(\tau)$, where τ is a simplex in $\mathcal{G}(X)$.

For example, let X be $\{30\}$, so the set of primes P which divide elements of X is $\{2, 3, 5\}$. Now there is no element of $\mathcal{G}(X)$ which maps to the vertex 2 in $\mathcal{D}(X)$, but $\eta(30)$ is the 2-simplex between 2, 3, and 5 which contains the vertex 2.

The following appears as a theorem in [3]

Theorem 8. *The common divisor simplicial complex on a set X is connected if and only if the prime divisor simplicial complex on X is connected.*

From now on we will assume that our simplicial complexes are connected, because the fundamental group of a simplicial complex is only defined on connected spaces.

Definition 9. The **2-skeleton** of a simplicial complex $\mathcal{K}$, denoted $\mathcal{K}^2$, is the union of all its simplices of dimension 2 or less. [7]

Given a simplicial complex $\mathcal{K}$, $\mathcal{K}^2$ is a new simplicial complex similar to $\mathcal{K}$ but with all 3-simplices and higher removed. The 2-dimensional faces of the higher dimensional simplices, which were removed, remain.

Using the concept of a 2-skeleton, we will discuss some nice properties of the fundamental group of a simplicial complex.

Theorem 10. *The fundamental group of a simplicial complex depends only on its 2-skeleton.*

Proof. Given a simplicial complex $\mathcal{K}$, we will show an isomorphism between equivalence classes of arbitrary loops in $\mathcal{K}$ and loops in $\mathcal{K}^2$. To do this, we need to map loops in $\mathcal{K}$ to loops in $\mathcal{K}^2$ and we need to show that this map is well-defined, a homomorphism, a surjection, and an injection. We will use a map that takes a loop in $\mathcal{K}$ to any homotopic loop in $\mathcal{K}^2$.

Let $\mathcal{L}$ be any loop through $\mathcal{K}$. Because all simplices are simply-connected, this path is homotopic to some path in $\mathcal{K}^2$. So $\mathcal{L}$, which is in one equivalence class in $\pi_1(\mathcal{K})$, is also in at least one equivalence class in $\pi_1(\mathcal{K}^2)$.

To show that this map is well-defined, we will show that all deformations of $\mathcal{L}$ to $\mathcal{K}^2$ will be in the same equivalence class in $\pi_1(\mathcal{K}^2)$. If $\mathcal{L}$ passes through a simplex of dimension 2 or lower, we are done – this simplex is in the 2-skeleton. On the other hand, if $\mathcal{L}$ passes through a simplex of dimension 3 or higher, call it σ, the 2-dimensional faces of σ will necessarily be in the 2-skeleton. From [5], these 2-dimensional faces will be homeomorphic to the surface of an n-sphere, which only has one equivalence class of loops. Therefore $\mathcal{L}$ can be deformed to one and only one equivalence class of loops on the 2-skeleton.

To see why this map is a surjection, note that every path in $\mathcal{K}^2$ is also in $\mathcal{K}$, so the identity homotopy will take any loop in $\mathcal{K}^2$ to itself. To show injection, let $\mathcal{L}$ be a loop in $\mathcal{K}^2$ and $\mathcal{L}_1$ and $\mathcal{L}_2$ in $\mathcal{K}$ homotopic to $\mathcal{L}$. Since these two are homotopic to $\mathcal{L}$, the homotopy map from $\mathcal{L}_1$ to $\mathcal{L}$ composed with the inverse of the homotopy map from $\mathcal{L}_2$ to $\mathcal{L}$ is a homotopy map from $\mathcal{L}_1$ to $\mathcal{L}_2$, so $\mathcal{L}_1$ and $\mathcal{L}_2$ are homotopic and we have an injection.

Thus the fundamental group of the simplicial complex is isomorphic to the fundamental group of the 2-skeleton of the simplicial complex. □

Definition 11. If $\mathcal{K}$ is a connected simplicial complex, an **edge-path** is a chain of vertices connected by edges in $\mathcal{K}$. An **edge-loop** is an edge-path starting and ending at the same vertex.

We will define a notation for an edge-path. Let $v_1, v_2, \ldots, v_n$ be a sequence of vertices with v_i and v_{i+1} connected by edges for i from 1 to $n-1$. For convenience when describing edges, let the edge between v_i and v_{i+1} be denoted by $\langle v_i, v_{i+1} \rangle$. Let the path along these vertices be denoted as $\langle v_1, v_2 \rangle \langle v_2, v_3 \rangle, \ldots, \langle v_{n-1}, v_n \rangle$. Note that using this notation, the edge-path is an edge-loop if and only if $v_1 = v_n$.

Theorem 12. *Every loop in a simplicial complex is homotopic to an edge-loop.*

Proof. By Theorem 10, we know that the fundamental group of a simplicial complex only depends on its 2-skeleton, so every loop in the complex is homotopic to a loop in the 2-skeleton. Now consider a loop that passes through a 2-simplex. Because simplices are simply-connected, this loop can be continuously deformed to the edge of the 2-simplex. □

Therefore the group of equivalence classes of edge-loops in the 2-skeleton is isomorphic to the fundamental group of the simplicial complex. Note that the fundamental group of the simplicial complex is not isomorphic to the fundamental group of the 1-skeleton. To see this, consider a simplicial complex consisting only of a 2-simplex. The fundamental group of the 2-simplex is the trivial group, whereas the fundamental group of the 1-skeleton of the 2-simplex is $\mathbb{Z}$.

Definition 13. A **simple equivalence** is an equivalence relation between edge-paths. Let v_1, v_2, and v_3 be three not necessarily distinct vertices in the simplicial complex, let P_1 be a path ending at v_1 and let P_2 be a path beginning at v_2. We define the path $P_1 \langle v_1, v_3 \rangle \langle v_3, v_2 \rangle P_2$ to be simply equivalent to the path $P_1 \langle v_1, v_2 \rangle P_2$ if and only if there exists a simplex containing all three of v_1, v_2, and v_3.

Note that the idea of a simple equivalence can take three forms, depending on which of v_1, v_2, and v_3 are distinct. If the three vertices are distinct, the simple equivalence takes the path $P_1 \langle v_1, v_3 \rangle \langle v_3, v_2 \rangle P_2$ to the path $P_1 \langle v_1, v_2 \rangle P_2$ if and only if there exists a 2-simplex between the vertices v_1, v_2, and v_3.

If $v_1 = v_2$, edge-paths of the form $P_1 \langle v_1, v_3 \rangle \langle v_3, v_1 \rangle P_2$ and $P_1 P_2$ are always simply equivalent, because by the definition of an edge-path there is a 1-simplex between v_1 and v_3.

If $v_1 = v_3$, $v_2 = v_3$, or $v_1 = v_2 = v_3$, the two edge-paths that are simply equivalent will look the same – the only difference is that repeated vertices in the path are removed.

Two edge-loops belong to the same equivalence class if one can be obtained from the other by a finite number of simple equivalences. With these equivalence classes we can define a group of equivalence classes of edge-loops, which is isomorphic to the fundamental group.

We will now show that η induces a map on edge-paths. But first we need a theorem regarding a very interesting subset reversal property of η.

Theorem 14. *(Subset Reversal) Given two simplices σ_1 and $\sigma_2 \in \mathcal{G}(X)$, if σ_1 is a face of σ_2, then $\eta(\sigma_2)$ is a face of $\eta(\sigma_1)$.*

Proof. Because σ_1 is a face of σ_2, its set of vertices are a subset of the set of vertices of σ_2. Therefore the set of primes in the greatest common divisor of the vertices of σ_2 are a subset of the set of primes in the greatest common divisor of the vertices of σ_1. This means that the dimension of $\eta(\sigma_2)$ is less than or equal to the dimension of $\eta(\sigma_1)$, and because the prime vertices in $\eta(\sigma_2)$ are also vertices of $\eta(\sigma_1)$ we have that $\eta(\sigma_2)$ is a face of $\eta(\sigma_1)$. $\square$

4 The Induced Map

We will show that this map induces a map η_* from edge-paths in $\mathcal{G}(X)$ to edge-paths in $\mathcal{D}(X)$. For edge-paths, we are only concerned with η's effect on 0- and 1-simplices. Because an edge-path consists of an alternating sequence of vertices and edges, we can look at the effect that η has on vertices and edges. By Theorem 14, the image of an edge is a face of the images of the vertices on either side of the edge, so we can simply look at the image of the edges in the path.

Definition 15. Let $L = \langle v_1, v_2\rangle \dots \langle v_{n-1} v_n\rangle$ be an edge-path in $\mathcal{G}(X)$. We define $\eta_*(\mathcal{L})$ as follows.

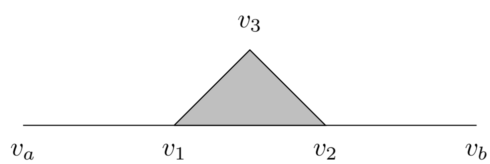
1. Fix a point $\omega_i \in \eta(v_i)$ for each $1 \le i \le n-1$ such that $\omega_i \in \langle v_1, v_{i+1}\rangle$.
2. Let p_i be a path from ω_i to ω_{i+1} for each $1 \le i \le n-1$ with all vertices in p_i in $\eta(v_{i+1})$.
3. Let $\eta_*(\mathcal{L})$ be the concatenation of all p_i for $1 \le i \le n-1$.

Note that all possible choices for p_i are homotopic because they are completely inside a simplex, and simplices are simply-connected.

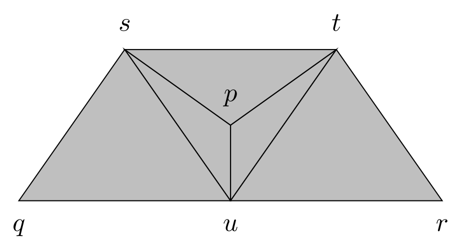
Theorem 16. *The induced map η_* is a well-defined map on edge-paths.*

Proof. Let $\mathcal{L}_0$ and $\mathcal{L}_1$ be two homotopic paths in $\mathcal{G}(X)$. We will show that their images are homotopic in $\mathcal{D}(X)$. For $\mathcal{L}_0$ and $\mathcal{L}_1$ to be homotopic in the domain means that there is a sequence of simple equivalences that maps $\mathcal{L}_0$

onto $\mathcal{L}_1$. Assume $\mathcal{L}_0$ and $\mathcal{L}_1$ differ by one simple equivalence. This means that given not necessarily distinct v_1, v_2, and v_3 in the edge-path, a simplex exists with v_1, v_2, and v_3 as vertices. For v_1, v_2, and v_3 to be part of a simplex in $\mathcal{G}(X)$, they must share a prime divisor which we will call p. Let $\mathcal{L}_0$ contain the edge-path segment $\langle v_a, v_1\rangle\langle v_1, v_2\rangle\langle v_2, v_b\rangle$ and $\mathcal{L}_1$ contain the edge-path segment $\langle v_a, v_1\rangle\langle v_1, v_3\rangle\langle v_3, v_2\rangle\langle v_2, v_b\rangle$. For every edge in $\mathcal{G}(X)$, we know that there exists a prime dividing the vertices on either side of the edge. Let q divide v_a and v_1, u divide v_1 and v_2, s divide v_1 and v_3, t divide v_3 and v_2, and r divide v_2 and v_b. We see that $\eta_*(\mathcal{L}_0)$ passes through the simplices $\eta(\langle v_a, v_1\rangle)$, $\eta(\langle v_1, v_2\rangle)$, and $\eta(\langle v_2, v_b\rangle)$. Because simplices are simply-connected any path through these simplices is homotopic to the path $\langle q, u\rangle\langle u, r\rangle$ in $\mathcal{D}(X)$. Likewise $\eta_*(\mathcal{L}_1)$ passes through $\eta(\langle v_a, v_1\rangle)$, $\eta(\langle v_1, v_3\rangle)$, $\eta(\langle v_3, v_2\rangle)$, and $\eta(\langle v_2, v_b\rangle)$, and is homotopic to the path $\langle q, s\rangle\langle s, t\rangle\langle t, r\rangle$.



(a) A Picture of $\mathcal{G}(X)$



(b) The Corresponding $\mathcal{D}(X)$

Our goal now is to show that because $\mathcal{L}_0$ and $\mathcal{L}_1$ are homotopic, the images of $\mathcal{L}_0$ and $\mathcal{L}_1$, $\langle q, u\rangle\langle u, r\rangle$ and $\langle q, s\rangle\langle s, t\rangle\langle t, r\rangle$, are homotopic. Now because q, u, and s all divide v_1, these three primes form a 2-simplex in $\mathcal{D}(X)$. Likewise since r, u, and t all divide v_2, these three primes also form a 2-simplex. In the same way because s, u, and p all divide v_1, t, u, and p all divide v_2, and s, t, and p all divide v_3, these three groups of three primes form three 2-simplices in $\mathcal{D}(X)$. Because we have all of these 2-simplices, we see that simple equivalences exist between

$$\langle q, s\rangle\langle s, t\rangle\langle t, r\rangle \text{ and } \langle q, s\rangle\langle s, p\rangle\langle p, t\rangle\langle t, r\rangle \text{ and } \langle q, s\rangle\langle s, u\rangle\langle u, p\rangle\langle p, t\rangle\langle t, r\rangle \text{ and}$$
$$\langle q, s\rangle\langle s, u\rangle\langle u, t\rangle\langle t, r\rangle \text{ and } \langle q, u\rangle\langle u, t\rangle\langle t, r\rangle \text{ and } \langle q, u\rangle\langle u, r\rangle.$$

Thus $\eta_*(\mathcal{L}_0)$ is homotopic to $\eta_*(\mathcal{L}_1)$.

Likewise if $\mathcal{L}_0$ and $\mathcal{L}_1$ differ by a finite number of successive simple equivalences, we can find a chain of intermediary paths from $\mathcal{L}_0$ and $\mathcal{L}_1$, each differing by a simple equivalence, with each path in the chain homotopic to the last.

Since each image in the chain is homotopic to the image of the last, we know that $\eta_*(\mathcal{L}_0)$ and $\eta_*(\mathcal{L}_1)$ are homotopic by transitivity. Therefore the induced map η_* is well-defined. □

Now we must show that the induced map η_* is a homomorphism on the fundamental groups; that is, η_* is operation preserving.

Theorem 17. *The map η_* is a homomorphism between $\pi_1(\mathcal{G}(X))$ and $\pi_1(\mathcal{D}(X))$.*

Proof. Let $\mathcal{L}_0$ and $\mathcal{L}_1$ be paths in $\mathcal{G}(X)$. Recall that the operation of the fundamental group is concatenation of paths, and that paths are continuous functions from $[0,1]$ to the topological space; that is,

$$\mathcal{L}_0 \circ \mathcal{L}_1 = \begin{cases} \mathcal{L}_0(2t) & : t \in [0, \frac{1}{2}] \\ \mathcal{L}_1(2t-1) & : t \in [\frac{1}{2}, 1] \end{cases} .$$

Now

$$\eta_*(\mathcal{L}_0 \circ \mathcal{L}_1) = \begin{cases} \eta_*(\mathcal{L}_0(2t)) & : t \in [0, \frac{1}{2}] \\ \eta_*(\mathcal{L}_1(2t-1)) & : t \in [\frac{1}{2}, 1] \end{cases} = \eta_*(\mathcal{L}_0) \circ \eta_*(\mathcal{L}_1),$$

so η_* is a group homomorphism between $\pi_1(\mathcal{G}(X))$ and $\pi_1(\mathcal{D}(X))$. □

Theorem 18. *The map η_*, as a homomorphism, is a surjection.*

Proof. Let $\mathcal{L}_0$ be an edge-loop in $\mathcal{D}(X)$. We will show that there exists a path in $\mathcal{G}(X)$ that maps to an edge-loop homotopic to $\mathcal{L}_0$ under η_*. We can write $\mathcal{L}_0$ as $\langle p_1, p_2\rangle\langle p_2, p_3\rangle \dots \langle p_{n-1}, p_n\rangle$ where $p_n = p_1$, and edges exist between successive vertices. There exist simplices $\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_n \in \mathcal{G}(X)$ such that p_i divides all of the vertices of σ_i for all i. Now given any σ_i and σ_{i+1} in this chain, because p_i and p_{i+1} are connected in $\mathcal{D}(X)$, there must be a simplex which all of its vertices are divisible by both p_i and p_{i+1}, which means that this simplex is a face of both σ_i and σ_{i+1}. Because they share a face, σ_i and σ_{i+1} are connected and we can define an edge-path $\mathcal{L}_1$ through these simplices. Because σ_1 is the same as σ_n this is an edge-loop. Then η_* applied to the path through σ_i and σ_{i+1} is a simplex containing p_{i+1} because p_{i+1} divides the vertices of both σ_i and σ_{i+1}, and $\eta_*(\mathcal{L}_1)$ is homotopic to $\mathcal{L}_0$. □

Theorem 19. *The map η_* is an isomorphism from $\pi_1(\mathcal{G}(X))$ to $\pi_1(\mathcal{D}(X))$.*

Proof. Because η_* is onto, it suffices to show that the kernel of η_* is trivial by Theorem 10.2.9 in [1]. We will proceed by induction.

We will show that if $\mathcal{L}$ is a loop in $\mathcal{D}(X)$ which is homotopic to the trivial loop, then the inverse image of that loop under η_* is also homotopic to the trivial loop. As a base case, let $\mathcal{L}$ be the edge-loop $\langle p, q\rangle\langle q, r\rangle\langle r, p\rangle$ in $\mathcal{D}(X)$, where p, q, and r are not necessarily distinct, and suppose the simplex with vertices p, q, and r exists. Because this simplex in $\mathcal{D}(X)$ exists, a simplex $\sigma_{\{p,q,r\}}$ in which every vertex is divisible by all members of the set $\{p, q, r\}$ exists in $\mathcal{G}(X)$. Likewise the simplices σ_p, where all vertices are divisible by p, σ_q, and σ_r exist.

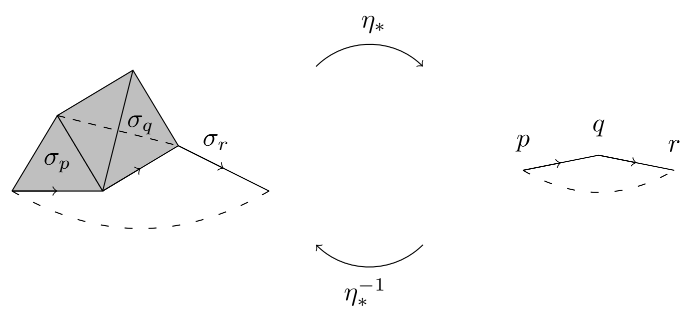


Figure 2: Illustration of Surjection Proof

Now let $\hat{\mathcal{L}}$ be a loop that begins at a point in σ_p, passes through σ_q and σ_r, and ends at the same point in σ_p. Now the image of $\hat{\mathcal{L}}$ under η_* is homotopic to $\mathcal{L}$. Because $\sigma_{\{p,q,r\}}$ exists and is a subset of all three of σ_p, σ_q, and σ_r, we can find a series of simple equivalences from $\hat{\mathcal{L}}$ to the trivial loop.

As our induction hypothesis, assume that given a loop $\mathcal{L}_0$ in $\mathcal{D}(X)$ which is homotopic to the trivial loop by K simple equivalences, the inverse image of $\mathcal{L}_0$ in $\mathcal{G}(X)$ is homotopic to the trivial loop. Let $\mathcal{L}_0$ be an edge-loop in $\mathcal{D}(X)$ which is homotopic to the trivial loop by K simple equivalences. Let $\mathcal{L}_1$ be any edge-loop which is simply equivalent to $\mathcal{L}_0$, which implies that $\mathcal{L}_0$ and $\mathcal{L}_1$ are homotopic, and $\mathcal{L}_1$ is homotopic to the trivial loop by $K+1$ simple equivalences. We can write $\mathcal{L}_0$ as the edge-loop $\langle x_0, x_1\rangle \ldots \langle p, r\rangle \ldots \langle x_n, x_0\rangle$ and $\mathcal{L}_1$ as $\langle x_0, x_1\rangle \ldots \langle p, q\rangle\langle q, r\rangle \ldots \langle x_n, x_0\rangle$ where p, q, and r are not necessarily distinct but there exists a simplex between them. Because the three are part of a simplex, in $\mathcal{G}(X)$ there exist simplices σ_p in which every vertex is divisible by p, σ_q divisible by q, σ_r divisible by r, and $\sigma_{\{p,q,r\}}$ divisible by p, q, and r. By Theorem 14, $\sigma_{\{p,q,r\}}$ is a face of all three of σ_p, σ_q, and σ_r. By our induction hypothesis, we know that there exists $\hat{\mathcal{L}}_0$ in $\mathcal{G}(X)$ such that $\hat{\mathcal{L}}_0$ is the inverse image of $\mathcal{L}_0$ under η_* and $\hat{\mathcal{L}}_0$ is homotopic to the trivial loop. Now we must show that the inverse image of $\mathcal{L}_1$, which we will denote $\hat{\mathcal{L}}_1$, is homotopic to $\hat{\mathcal{L}}_0$. We know that $\hat{\mathcal{L}}_0$ passes through σ_p and σ_q. We also know that $\hat{\mathcal{L}}_1$ passes through σ_p, σ_r, and σ_q. But because σ_p, σ_q, and σ_r share the face $\sigma_{\{p,q,r\}}$, we know that we can form a series of simple equivalences between $\hat{\mathcal{L}}_0$ and $\hat{\mathcal{L}}_1$, and thus from $\hat{\mathcal{L}}_1$ to the trivial loop. □

Thus for any set of integers, the common and the prime divisor simplicial complexes are isomorphic.

5 Conclusion and Further Studies

We know that the common and the prime divisor simplicial complex have isomorphic fundamental groups, but can this be extended? Could this be proven using only relationships between numbers, like the relationship between common and prime divisors, not simplicial complexes at all? If so, can we find other relationships between numbers that exhibit the same isomorphic fundamental group property?

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