Mathematical Models of Conventional Warfare

Tod O. Schulter
Carthage College
tschulter@carthage.edu

January 30, 2016

Abstract

Modern conventional warfare consists of multiple forms of battle that can be mathematically modeled using differential equations. One such basic and widely analyzed system of differential equations is the Lanchester Model. In this report we define multiple general cases of the basic Lanchester Model and explore a specific example of each highlighted case. Through computation and numerical integration we then obtain solutions for each of these cases and construct plots and tables to visualize and demonstrate these results. From this information, we lastly draw certain practical conclusions about the specific Lanchester Models in this report and within what criteria they would best be employed both by strategists and on the battlefield.

1 Introduction

Throughout history, the tools men have utilized in times of war have changed dramatically, while the method by which we have waged war has remained essentially the same. This method is called conventional warfare. To aid in generating victorious battle strategies in the time surrounding the World Wars, an English engineer and polymath named Frederick W. Lanchester developed a system of differential equations (DE) called, in recognition of their creator, the Lanchester Models. These equations have been utilized throughout conventional military combat situations in the form of operations research to better plan for and predict battles and their potential outcomes. Additionally, these equations have been manipulated to apply to multiple battle scenarios such as aimed fire, area fire, and ancient combat battles. In this report, we examine the Lanchester models for these situations and the outcomes therein generated to ultimately determine which of the inspected systems and battle styles would theoretically be the most effective for battle strategists to employ both inside and outside the battlefield.

2 Definitions and Development

2.1 Definitions

Before beginning this examination, we must first present basic definitions and ideas relevant to the subsequent calculations.

Definition 1. Conventional warfare is defined as battle between two forces with conventional means, i.e. typical fighting implements, with the exclusion of any chemical, biological, or nuclear weapons.

In conventional warfare situations, the opposing forces must directly interact with one another. Mathematically, this can lead to multiple intriguing situations, as well as a higher possibility of predicting specific battle outcomes and casualties. Non-conventional warfare does not allow for this level of dual interaction, and such is our reason for focusing on the chosen forms of conventional warfare in this mathematical report.

Definition 2. The **Lanchester Model** is a system of DEs describing the time dependence of attacker and defender troop sizes, $x(t)$ and $y(t)$ respectively, as functions of time which only depend on the relative attacker and defender strengths [1].

For instance, a generalized basic Lanchester Model could appear as

$$\begin{aligned} \frac{dx}{dt} &= -S_y\, y(t) \\ \frac{dy}{dt} &= -S_x\, x(t) \end{aligned} \tag{1}$$

where S_x and S_y are the **fighting effectiveness coefficients (FEC)**, with units of inverse time, of the attackers and defenders respectively [2]. Simply, these values are numerical representations of how well each respective force fights. Note also that there is no defined difference between the attacking and defending forces, and thus without loss to generality we herein assign $x(t)$ as the attacker and $y(t)$ as the defender for the reminder of this study. As an aside, we also must define our time variable t in days, as when comparing all the obtained results and their values of t to measurements of time, we found days to be the most reasonable. This will become more evident during the Results section of the report. Continuing, it is also relevant to define and give generic examples of the specific forms of Lanchester Models that will be examined.

Definition 3. An **aimed fire** battle is one where the attacking and defending sides specifically aim at and attack each other. For context, an example of such would be a typical Revolutionary War battle. An example of the associated Lanchester Model can be seen in Equation (1), as the change in the size of each force is affected only by the FEC and size of the opposing force, as would be expected for this style of battle.

Definition 4. An **ancient combat** type of battle is one where the attacking and defending forces engage in hand-to-hand combat, possibly with knives, swords, or other personal weapons. An example Lanchester Model [2] for this battle style can be seen in Equation (2), where only the fighting effectiveness of each side cause a change in the quantity of troops of the other:

$$\begin{aligned} \frac{dx}{dt} &= -S_y \\ \frac{dy}{dt} &= -S_x. \end{aligned} \tag{2}$$

Definition 5. An **area fire** battle is one where the two opposing forces attack the general area the opposite force is occupying, such as in an artillery barrage. Equation (3) gives the associated Lanchester Model [2]:

$$\begin{aligned} \frac{dx}{dt} &= -S_y\, y(t)\, x(t) \\ \frac{dy}{dt} &= -S_x\, x(t)\, y(t). \end{aligned} \tag{3}$$

Finally, in order to solve these systems of equations, we will use the following technique.

Definition 6. The **Laplace transform** function Y of the function y is defined by

$$Y(s) = \int_0^\infty y(t)\, e^{-st}\, dt$$

for all numbers s for which this improper integral converges. [3]

Example 7. Let $y(t) = \sinh \alpha t$ where $\alpha \in \mathbb{R}$, and assume $\alpha < s$. The Laplace transform of this function is as follows:

$$\begin{aligned} Y(s) &= \int_0^\infty y(t)\, e^{-st}\, dt \\ &= \int_0^\infty \sinh \alpha t\, e^{-st}\, dt \\ &= \int_0^\infty \frac{1}{2}\left(e^{\alpha t} - e^{-\alpha t}\right) e^{-st}\, dt \\ &= \lim_{x\to\infty} \frac{1}{2}\left[\left(\frac{1}{\alpha - s}\right) e^{(\alpha - s)t} + \left(\frac{1}{\alpha + s}\right) e^{-(\alpha+s)t}\right]\Bigg|_0^x \\ &= -\frac{1}{2}\left(\frac{1}{\alpha - s} + \frac{1}{\alpha + s}\right) \\ &= \frac{\alpha}{s^2 - \alpha^2}. \end{aligned}$$

Thus the Laplace transform for $\sinh \alpha t$ is $\frac{\alpha}{s^2-\alpha^2}$. Similarly we follow this same procedure for the transform of $\cosh \alpha t$, giving $\frac{s}{s^2-\alpha^2}$. We present these results in Table 1.

Function t-Domain	**Laplace s-Domain**
$\sinh \alpha t$	$\frac{\alpha}{s^2-\alpha^2}$
$\cosh \alpha t$	$\frac{s}{s^2-\alpha^2}$

Table 1: Laplace Hyperbolic Trigonometric Substitutions for $\alpha < s$ and $\alpha, s \in \mathbb{R}$.

Also required with each of the above example Lanchester Models are the initial quantities of troops on each side: the **initial conditions (IC)** for each DE. In accordance with mathematical terminology, we thus let $x(0) = x_0$ and $y(0) = y_0$. Now, we delve into specific Lanchester Model situational examples and their respective derivations.

2.2 Development

Here we begin by looking at a specific case of Equation (1) and the work involved in calculating its solutions [4].

Example 8. To begin, we first declare the initial troop quantities of the $x(t)$ and $y(t)$ forces, as well as the FECs for each force. Let $S_y = 3$ and $S_x = 2$, with ICs of $x_0 = 250$ and $y_0 = 400$. We now observe this model as such:

$$\frac{dx}{dt} = -3\,y(t), \quad x_0 = 250 \tag{4a}$$

$$\frac{dy}{dt} = -2\,x(t), \quad y_0 = 400. \tag{4b}$$

We first inspect the manual calculation of the solutions to this model, wherein we employ the DE technique of Laplace Transforms (see Definition 6), allowing us to write each of the model equations in terms of s and the anti-derivatives of $x(t)$ and $y(t)$, $X(t)$ and $Y(t)$ respectively. Applying this transform gives

$$sX(s) - 250 = -3\,Y(s)$$
$$sY(s) - 400 = -2\,X(s).$$

From these equations, we initially solve the second transformed equation for $X(s)$, giving

$$X(s) = \frac{-sY(s)}{2} + 200,$$

which we then insert into the first transformed equation in order to solve for $Y(s)$. This gives

$$Y(s) = \frac{500 - 400s}{-s^2 + 6}.$$

Next we employ some basic algebraic manipulation to this equation to rewrite the numerator terms separately, in the hope of finding a substitution to reinsert

t back into this equation in place of s. This results in

$$Y(s) = -500\left(\frac{1}{s^2-6}\right) + 400\left(\frac{s}{s^2-6}\right),$$

where we find the s terms resemble the Laplace s-domain functions as listed in Table 1. However, once we identify the hyperbolic cosine portion of our current equation, we also notice the hyperbolic sine s-domain function does not exactly match the remaining s portion of the equation. Therefore we multiply the numerator and denominator of this portion by $\alpha = \sqrt{6}$, as we see $\alpha^2 = 6$. Thus we transform back to the time domain and have

$$Y(s) = \frac{-500}{\sqrt{6}}\left(\frac{\sqrt{6}}{s^2-6}\right) + 400\left(\frac{s}{s^2-6}\right) \tag{5a}$$

$$\Rightarrow y(t) = \frac{-500}{\sqrt{6}}\left(\sinh(\sqrt{6}t)\right) + 400\left(\cosh(\sqrt{6}t)\right). \tag{5b}$$

From Equation (5b), we replace the hyperbolic trigonometry terms with their exponential equivalents and simplify to give the complete $y(t)$ equation:

$$y(t) = \left(\frac{-250}{\sqrt{6}} + 200\right)e^{\sqrt{6}t} + \left(\frac{250}{\sqrt{6}} + 200\right)e^{-\sqrt{6}t}. \tag{6}$$

To find the complete $x(t)$ equation, we look back and reference Equation (4b). This equation already has $x(t)$ within it, and with the $y(t)$ equation we just found we can solve for $x(t)$ by differentiating the $y(t)$ solution and setting it equal to Equation (4b). Thus we have our $x(t)$ equation as follows:

$$x(t) = -25\,e^{-\sqrt{6}t}(-5 - 4\sqrt{6} + (-5 + 4\sqrt{6})\,e^{2\sqrt{6}t}) \tag{7}$$

To this point, we have solved our first example Lanchester Model for its two solutions, equations representing the number of attacking $x(t)$ troops with respect to time and the number of defending $y(t)$ troops with respect to time; we investigate these results at a later point in this discussion. Prior to this, we construct Lanchester Models following the two remaining types of combat investigated, area fire and ancient combat, and identify the solutions to these models.

Example 9. We move first to an ancient combat example. Similarly to Example 8, we also need FECs for the attacking and defending forces, as well as their respective initial troop strengths. For the sake of consistency, we implement the same values used in that example; that is, $S_y = 3$, $S_x = 2$, $x_0 = 250$, and $y_0 = 400$. From Equation (2), we write this model of DEs as follows:

$$\frac{dx}{dt} = -3 \quad x_0 = 250$$
$$\frac{dy}{dt} = -2 \quad y_0 = 400$$

In solving this model for its respective $x(t)$ and $y(t)$ solutions, we follow the basic DE solution technique of integration and find that

$$\begin{aligned} x(t) &= -3t + c_1 \\ y(t) &= -2t + c_2, \end{aligned}$$

where we employ the ICs to solve for constants c_1 and c_2, finding

$$\begin{aligned} x(t) &= -3t + 250 \\ y(t) &= -2t + 400. \end{aligned} \tag{8}$$

Thus, we have the solutions for this ancient combat simulation.

Next, we move on to an example of the area fire mode of combat where the model is based from Equation (3).

Example 10. Once again, we establish our FECs and ICs in the same manner as Examples 8 and 9: let $S_y = 3$, $S_x = 2$, $x_0 = 250$, and $y_0 = 400$. Therefore, we write the model as

$$\begin{aligned} \frac{dx}{dt} &= -3\,x(t)\,y(t) \quad x_0 = 250 \\ \frac{dy}{dt} &= -2\,x(t)\,y(t) \quad y_0 = 400. \end{aligned}$$

In this instance, each DE contains both $x(t)$ and $y(t)$, and as such, computing a solution manually would be a difficult process to work through. Thus, to find the solutions for this model we used the computational software *Wolfram Mathematica* as shown in Appendix 1. This resulted in the equations for $x(t)$ and $y(t)$ displayed in Equation (9).

$$\begin{aligned} x(t) &= \frac{1750}{-5 + 12\,e^{700t}} \\ y(t) &= \frac{2800\,e^{700t}}{-5 + 12\,e^{700t}} \end{aligned} \tag{9}$$

Example 11. Lastly, we examined another battle scenario wherein the attacking and defending forces fought both with aimed fire and area fire strategies. To model this, we used a combination of the respective aimed and area fire models, Equations (1) and (3):

$$\frac{dx}{dt} = -S_{y_m}\,y(t) - S_{y_a}\,x(t)\,y(t) \tag{10a}$$

$$\frac{dy}{dt} = -S_{x_m}\,x(t) - S_{x_a}\,x(t)\,y(t). \tag{10b}$$

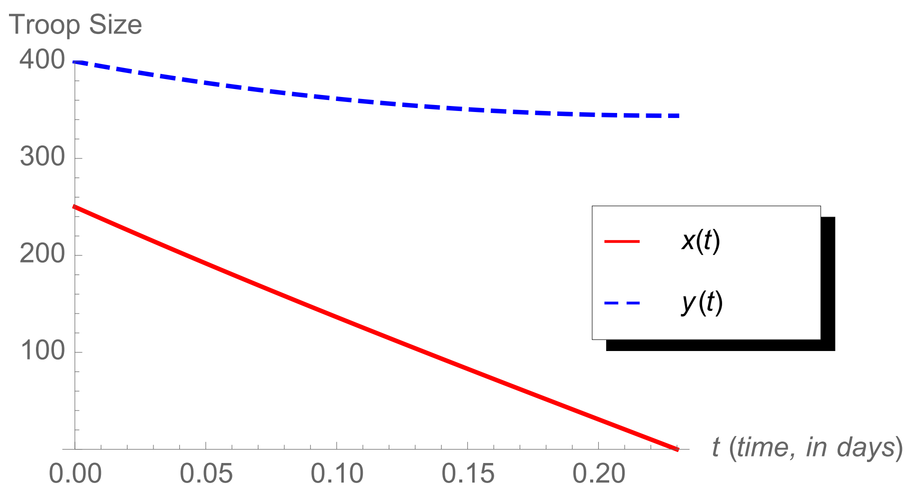


Figure 1: The initial plots of $x(t)$ and $y(t)$ for Equation (4), the Aimed Fire Example model

For this model, we employ four distinct FECs, two in each differential equation. In Equation (10a), we have S_{y_m} as the aimed fighting effectiveness coefficient and S_{y_a} as the area fighting effectiveness coefficient for the y-force. The FECs in Equation (10b) respectively represent the same quantities, except for the x-force. In assigning values to these coefficients, we followed the pattern of the previous examples and let the first coefficients in each equation, S_{y_m} and S_{x_m}, be 3 and 2 respectively. For the remaining FECs, we assigned $S_{y_a} = 2$ and $S_{x_a} = 3$, to ensure the solutions for each DE would be different from one another. Finally, we again let $x_0 = 250$ and $y_0 = 400$. With all these values identified, we proceeded to use *Mathematica* to solve this model, and while this did solve the model for numerical and graphical results, it could not determine closed-form solution equations. The graphical results obtained for this scenario however have been included in this report in the following Results section.

3 Results

3.1 Aimed Fire Model

We begin this section of the report by examining the graphical and numerical results obtained from the solutions to Example 8, Equations (6) and (7). These solutions are depicted in Figure 1, where we have plotted the size of each force versus the time over which the battle progresses. As would be expected from the FECs and ICs chosen, the defending $y(t)$ force is victorious, both through starting with a larger troop quantity and higher FEC compared to the attacking $x(t)$ force. It would take less than a quarter of a day for the defenders to defeat

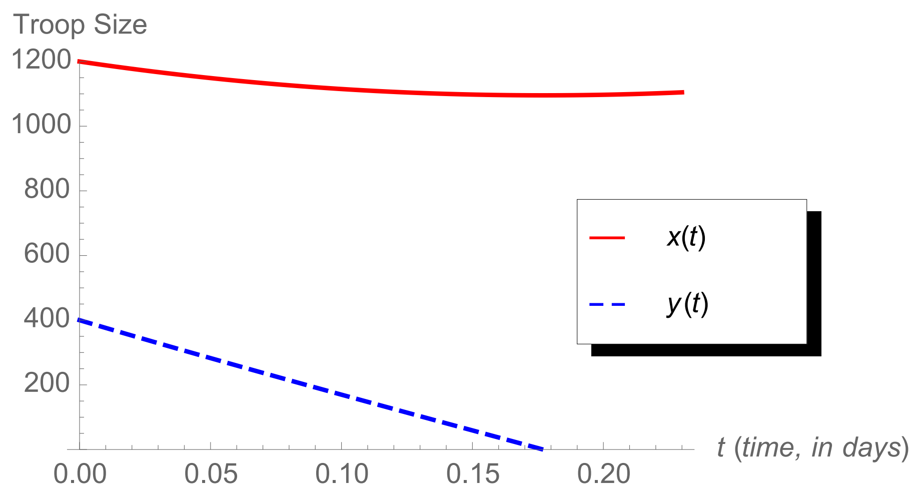


Figure 2: The plot of the solutions from the Aimed Fire Example with $x_0 = 1200$

Aimed Fire Results	**End Time (t_e)**	**ICs (x_0, y_0)**	**FECs (S_y, S_x)**	$x(t_e)$	$y(t_e)$
Figure 1 Values	0.2299	$(250, 400)$	$(3, 2)$	0	344
Figure 2 Values	0.1769	$(1200, 400)$	$(3, 2)$	1095	0
Figure 3 Values	0.3491	$(1200, 400)$	$(19, 2)$	0	92

Table 2: Final Aimed Fire troop values for x and y forces in various situations

the attackers in this scenario.

This result raises a new question though: if the FECs remained the same and the starting number of $x(t)$ troop was much higher, say three times the $y(t)$ population, how would the outcome of the battle differ? To answer this, we simply changed x_0 to 1200, and replotted the solutions in Figure 2. In this graph, we see the higher FEC of the y force was not great enough to overpower the significantly higher troop quantity of the x force. Thus, the defenders lost this battle with minimal impact on the attackers. Leaving these ICs as they are, we varied the FEC for the y force in an attempt to determine the minimum FEC value this force would need to defeat the much greater x force. After a few trials, we came to the conclusion that with $S_y = 19$, the y force would be victorious over the attacking x force (see Figure 3). Therefore, we found that in this case the smaller of the two forces would need a FEC of almost ten times that of the larger force to be victorious under these conditions. Lastly, we have included Table 2 to depict, roughly, the final troop quantities for each force in each of the three discussed scenarios.

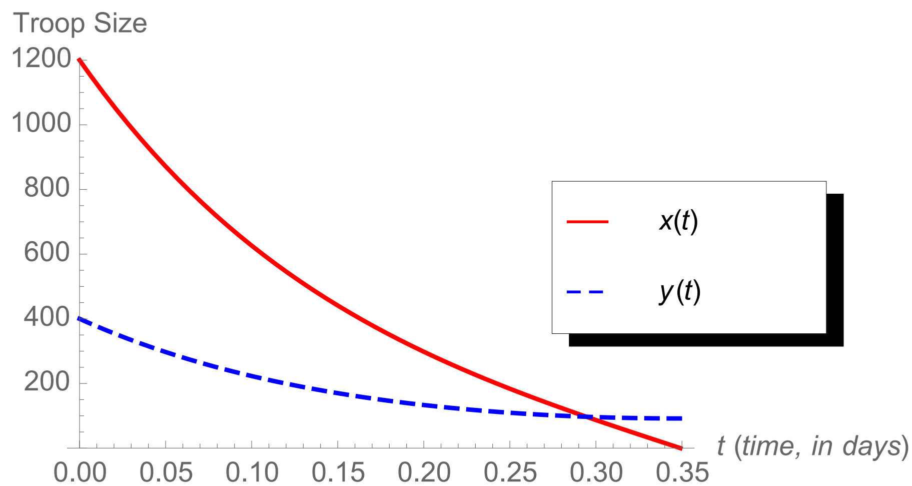


Figure 3: The plot of the solutions from the Aimed Fire Example with $x_0 = 1200$ and $S_y = 19$

3.2 Ancient Combat Model

Now we move to the results obtained from the solutions to Example 9. Again, after calculating these solutions, listed in Equation (8), we plotted them with respect to time in *Mathematica* and obtained Figure 4. As would be expected, these solutions are simple linear plots with respect to t. Again, with the FECs and ICs the same as the initial Aimed Fire model, we see that the y force easilly defeats the smaller and less-adept x force. Increasing the initial x troop quantity though to 1200 produces Figure 5. As in Figure 2, when the initial quantity of x troops is triple that of the y force, the x force is able to handily win the battle. To let the y force be victorious once again, we edited the FEC for the y force until we found a solution in which the x force lost. The value this occurred at was $S_y = 7$, and the resulting plot is given in Figure 6. For this type of warfare, we have found that when one force is three times the size of the other, the smaller of the two can be victorious when its fighting strength is almost four times that of the larger force. Once again, we include a table of the numerical results from these situations.

Lastly, a significant item of note is the amount of time t_e it took for one force to defeat the other in all the examined situations. When compared to the t_e values listed in Table 2, we see that those for this Ancient Combat scenario are much larger; thus, each proposed Ancient Combat battle would take a much longer amount of time to complete than any of the Aimed Fire battles, a plausible fact considering the tools of war that would be implemented in each respective type of combat. Also of note is the quantity of x troops remaining after defeating the y force when x_0 is 1200. In this case, we see there would

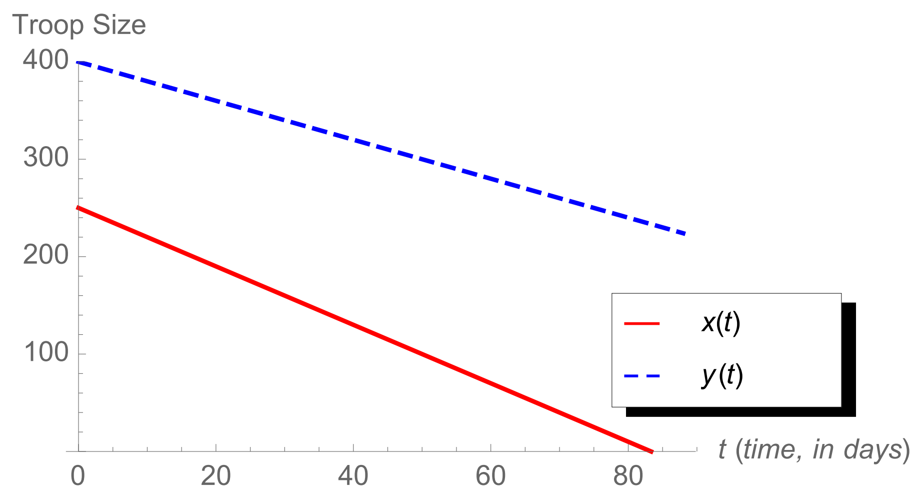


Figure 4: The initial plots of $x(t)$ and $y(t)$ for Example 9, the Ancient Combat Example

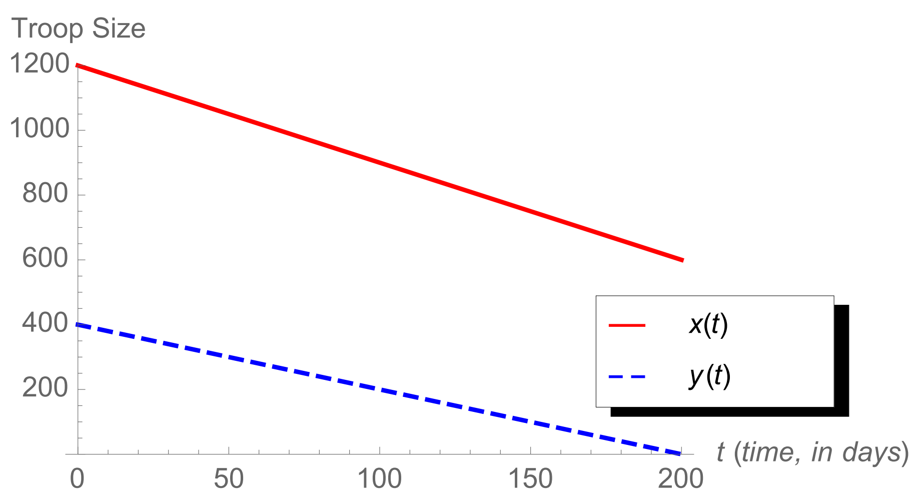


Figure 5: The plot of the solutions from the Ancient Combat Example with $x_0 = 1200$

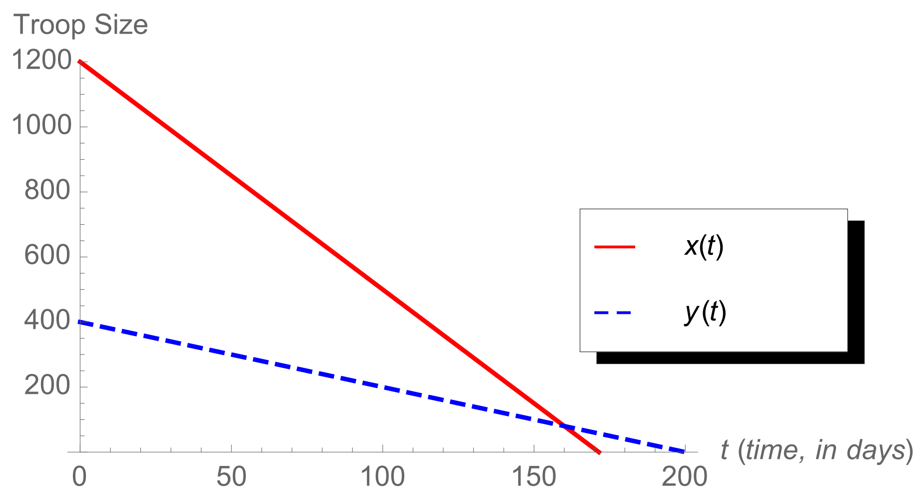


Figure 6: The plot of the solutions from the Ancient Combat Example with $x_0 = 1200$ and $S_y = 7$

Ancient Combat Results	**End Time (t_e)**	**ICs (x_0, y_0)**	**FECs (S_y, S_x)**	$x(t_e)$	$y(t_e)$
Figure 4 Values	83.33	$(250, 400)$	$(3, 2)$	0	233
Figure 5 Values	200	$(1200, 400)$	$(3, 2)$	600	0
Figure 6 Values	171.42	$(1200, 400)$	$(7, 2)$	0	57

Table 3: Final Ancient Combat troop values for x and y forces in various situations

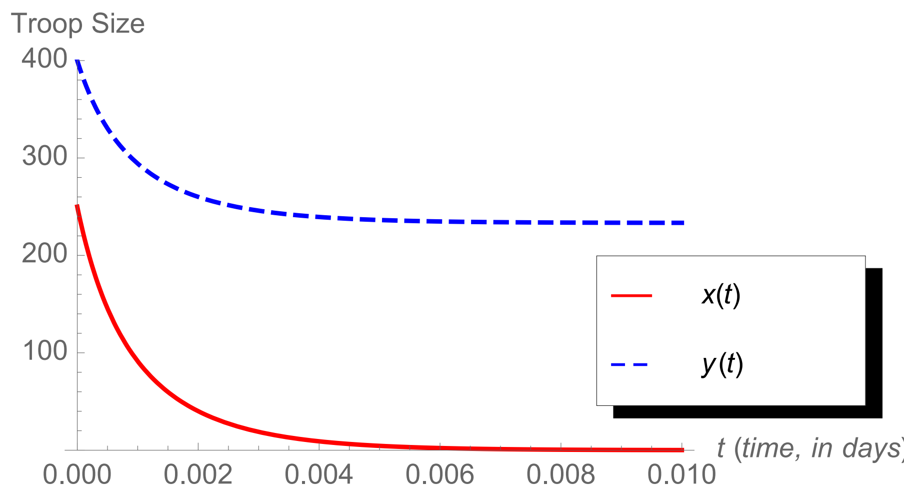


Figure 7: The initial plots of $x(t)$ and $y(t)$ for Example 10, the Area Fire Example

be only 600 troops remaining, whereas in the respective Aimed Fire scenario (Figure 2), there would be roughly 1095. This demonstrates that under these conditions, the Aimed Fire system of combat would be more beneficial for the attacking x force than Ancient Combat system would be; many more x troops would survive through an Aimed Fire battle versus an Ancient Combat scenario.

3.3 Area Fire Model

Next we examine the results of the Area Fire Model, discussed in Example 10. As we have done in the previous two subsections, we again plotted the solutions to the current example in *Mathematica* and have included them in Figure 7. Once again, the x force suffers a quick loss. Upon increasing the x force initial troop quantity to 1200, we observed the same result as within the previous two examples, that the y force now was the side to suffer the rapid loss. This is shown in Figure 8. Finally, we again changed the fighting strength of the y force such that under this form of combat the y force would once again defeat the x force despite its much greater size. Surprisingly, we found an FEC of $S_y = 7$, exactly the same as in the Ancient Combat scenario, would achieve this result. The plot representing this is shown in Figure 9. As with the discussions of the previous examples, a table of the numerical results for this example has also been included in Table 4. With these results determined, we have discovered another surprising correlation. When examining the x_{t_e} and y_{t_e} values for all cases of both the Ancient Combat scenario and the Area Fire scenario, we observe that the values are nearly identical. In the first case, the final y_{t_e} troop sizes are both approximately 234, and in the second case, the final x_{t_e} troop quantities

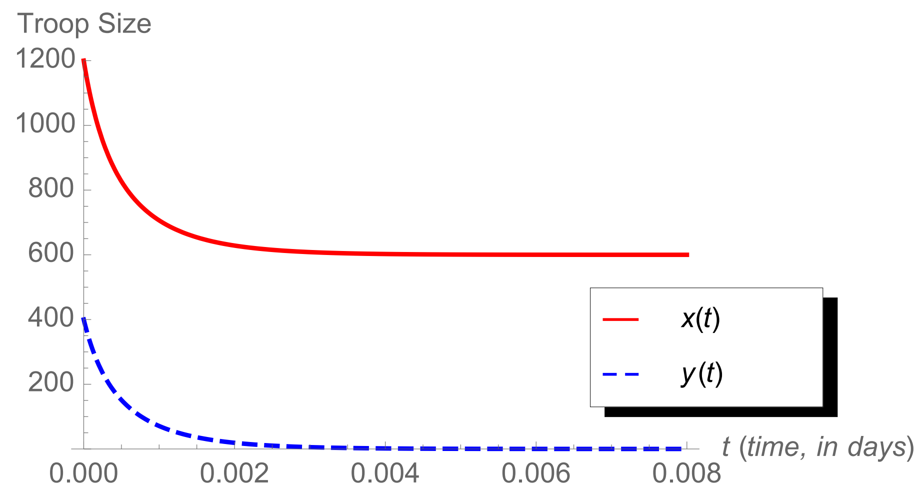


Figure 8: The plot of the solutions from the Area Fire Example with $x_0 = 1200$

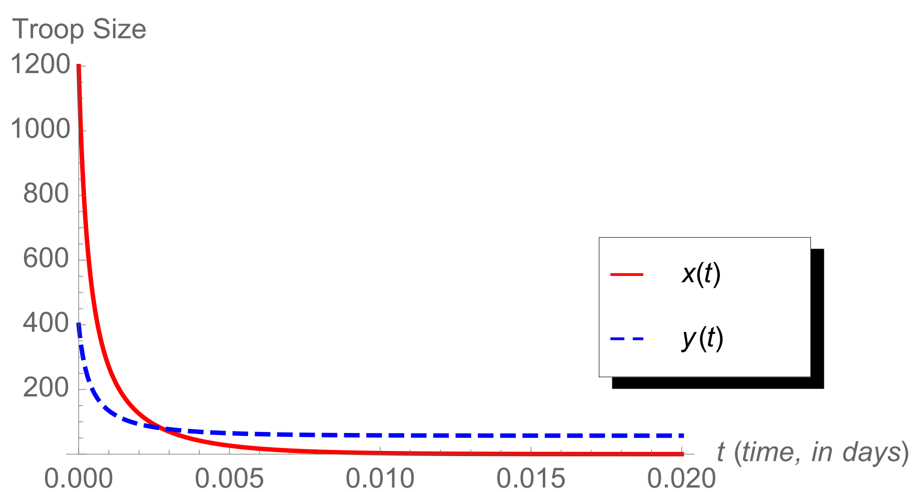


Figure 9: The plot of the solutions from the Area Fire Example with $x_0 = 1200$ and $S_y = 7$

Area Fire Results	End Time (t_e)	ICs (x_0, y_0)	FECs (S_y, S_x)	$x(t_e)$	$y(t_e)$
Figure 7 Values	0.009	$(250, 400)$	$(3, 2)$	0	234
Figure 8 Values	0.008	$(1200, 400)$	$(3, 2)$	600	0
Figure 9 Values	0.020	$(1200, 400)$	$(7, 2)$	0	57

Table 4: Final Area Fire troop values for x and y forces in various situations

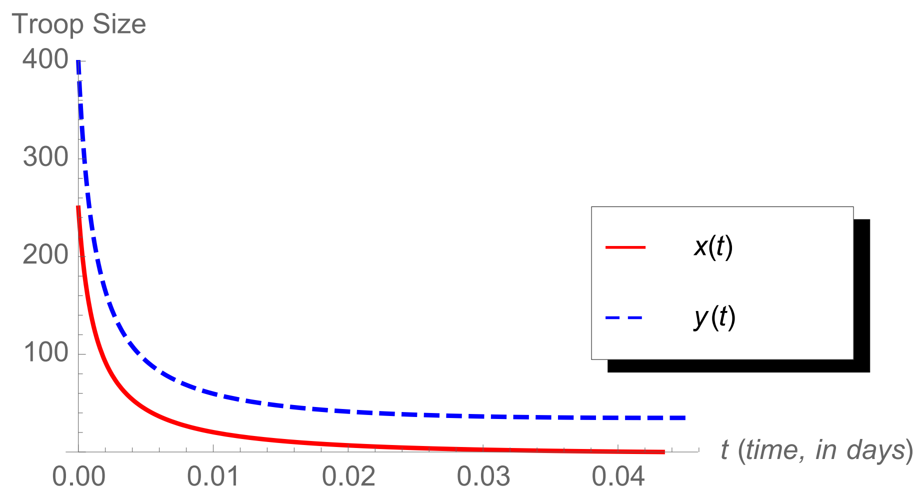


Figure 10: The initial plots of $x(t)$ and $y(t)$ for Equation (10), the Aimed and Area Fire Example model

are both approximately 600. In the third case, where the S_y value for both was also 7, the final y_{t_e} quantities were found to be about 57.16. Of drastic difference between these two models though was the final times t_e found in each of the three cases, with the t_e of the Ancient model much greater than the respective t_e values for the Area Fire model. Thus, we can draw the conclusion that while the Ancient and Area warfare methods can achieve roughly the same results given a certain scenario, the Area Fire method can achieve these results in a much shorter amount of time. For all intents and purposes, the Area Fire method is the modern equivalent of the Ancient Combat method.

3.4 Aimed and Area Fire Model

To conclude this section, we investigate the results obtained from the specific case of Equation (10), following mostly the same form as the previously inspected three cases. First, we include a plot of the solutions for the initial case discussed, given in Figure 10. Next, increasing the value for x_0 to 1200 in this

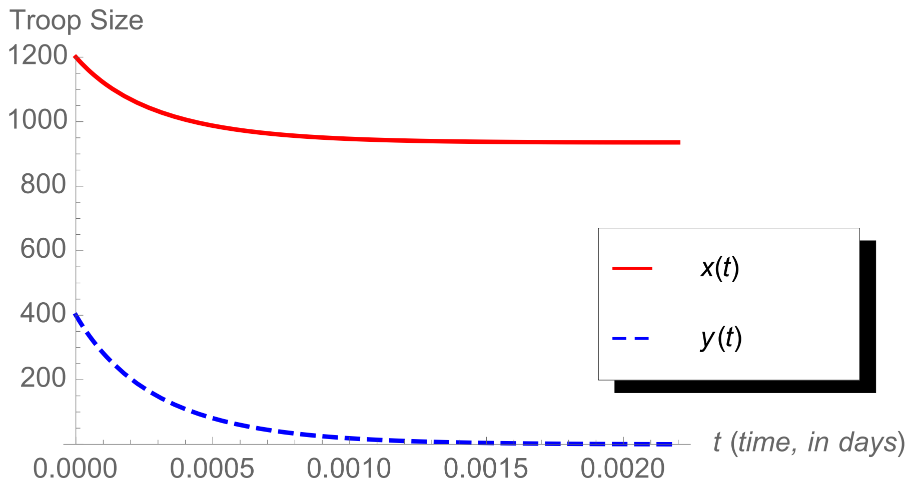


Figure 11: The plots of the solutions from the Aimed and Area Fire Example with $x_0 = 1200$

case produced Figure 11. From this point, our investigation method begins to differ from the precedent. Now, we investigated two rather than one alternate FEC case that would lead to victory for the y-force, the first being a higher S_{y_m} and the second being a raised S_{y_a}. In the first case, shown in Figure 12, we found the value for S_{y_m} would have to increase to 4000 in order to produce a victory for the y-force, with $S_{y_a} = 2$. In the second, shown in Figure 13, we found S_{y_a} would only have to increase to 11 to grant the y-force victory, with S_{y_m} back at 3. Also of note is that both of these instances resulted in a y victory in roughly the same amount of time, as shown in Table 5, despite the vastly different FEC values.

Lastly, we see from the solutions given in Table 5 for the $y(t_e)$ that when the area fire FEC is the deciding factor of the battle, there are more y troops surviving after the battle versus when the aimed fire FEC decides the battle. With all this evidence presented, we can conclude that the Area Fire mode of combat is the more efficient and less costly type of warfare in a similar battle scenario.

4 Conclusion and Directions for Further Research

After completing all the specified models and the example cases for each, as well as presenting the results obtained from these examples, we have made some interesting discoveries about the various Lanchester Models and various methods of conventional warfare. Through Example 8, we discovered that the Aimed Warfare method of combat can result in very low casualties for a force

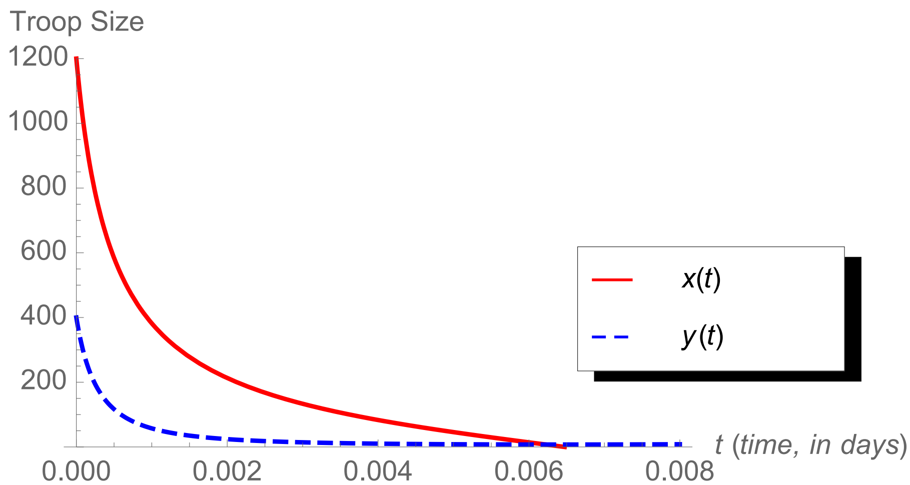


Figure 12: The plots of the solutions from the Aimed and Area Fire Example with $x_0 = 1200$ and $S_{y_m} = 4000$

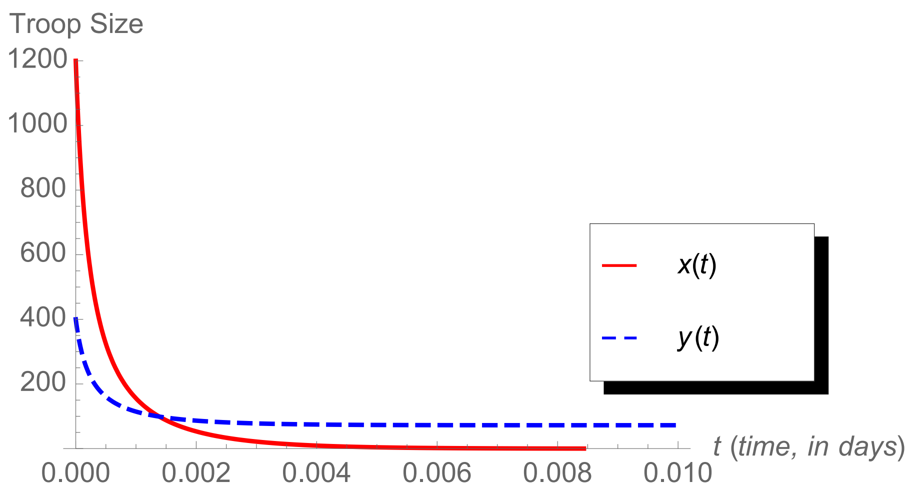


Figure 13: The plots of the solutions from the Aimed and Area Fire Example with $x_0 = 1200$ and $S_{y_a} = 11$

Aimed and Area Fire Results	**End Time (t_e)**	**ICs (x_0, y_0)**	**FECs (S_{y_m}, S_{y_a}); (S_{x_m}, S_{x_a})**	$x(t_e)$	$y(t_e)$
Figure 10 Values	0.043	$(250, 400)$	$(3, 2)$; $(2, 3)$	0	35
Figure 11 Values	0.00215	$(1200, 400)$	$(3, 2)$; $(2, 3)$	936	0
Figure 12 Values	0.00647	$(1200, 400)$	$(4000, 2)$; $(2, 3)$	0	7
Figure 13 Values	0.0081	$(1200, 400)$	$(3, 11)$; $(2, 3)$	0	72

Table 5: Final Aimed and Area Fire troop values for x and y forces in various situations

if that force is able to significantly overpower the opposing force, in our case by having three times as many troops as the opposing force. Also, we found that for the smaller of these two forces to overpower the larger, it would need a much higher fighting effectiveness coefficient than most of the other forms of combat. In Example 9, we found that the Ancient Combat method takes a much longer time for one side to achieve victory over the other, a fact that reinforces the "ancient" title for this form of combat. Through this example and Example 10, we also found that the Area Fire method is, for all intents and purposes, the modern form of Ancient Combat, as nearly identical results were achieved in both examples, with the only difference being the length of time for a battle to end. Lastly, through the combined Aimed and Area Fire example, we learned that a battle involving these two methods of warfare can be decided more favorably for one side if that side increases the fighting effectiveness of its area fire troops over that of its aimed fire troops. This FEC increase also can be more easily achieved than an increase for the aimed fire troops, as the value required for a favorable outcome through this method is much larger than that needed for an area fire victory.

Overall, in comparing all of these systems and results to determine the most favorable and effective model of modern warfare, we can conclude that the Area Fire method is the most effective in determining the outcome of a battle over the shortest amount of time, while the Aimed Fire method is the best at preserving a high troop quantity for the battle victor. These two methods are very similar to each other, and the Area Fire could outright be the more favorable of the two, if not for the smaller amount of surviving troops after a battle. In the end, the best method of combat to adopt in battle depends on whether a quick resolution or higher surviving troop quantity is more important to the warring forces.

For further research into this topic, more variations of the Lanchester Model could be inspected and solved, as well as more research into various initial condition and fighting effectiveness coefficient cases for both the models discussed within this report, and others not examined. Inspecting the outcomes and effects of battles with more than two warring forces could also lead to interesting insights and discoveries. Finally, other mathematical models of conventional warfare could be examined and compared with the Lanchester Model to determine which model or system would be the most advantageous for warring forces, and predicting the outcomes of potential combat situations.

References

[1] Jim Patras. *Lancaster Equations*, `http://lists101.his.com/pipermail/intelforum/2000-July/001909.html`. Accessed: 24 October 2015.

[2] *The Military Landscape: Mathematical Models of Combat*, Woodhead Publishing Limited, (1993).

[3] Glen R. Hall, Paul Blanchard, Robert L. Devaney. *Differential Equations*, Brooks/Cole, 4th edition, (2012).

[4] Prof. David Joyner. *An introduction to systems of DEs: Lanchester's equations for battle*, (2007), March.

[5] Wolfram Research, Inc., Mathematica, Version 10.0, Champaign, IL (2014).

[6] James G. Taylor. *Lanchester-Type Models of Warfare, Volume 1.* Naval Postgraduate School, (1980).

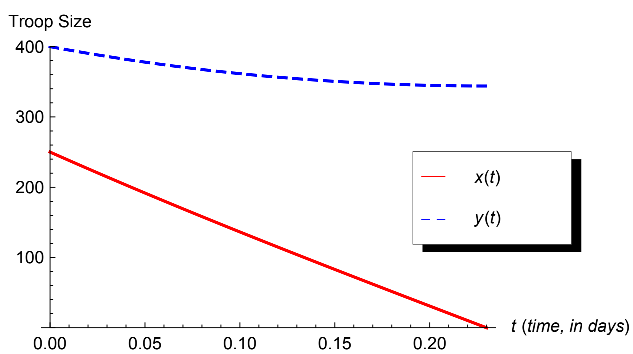
Appendix 1

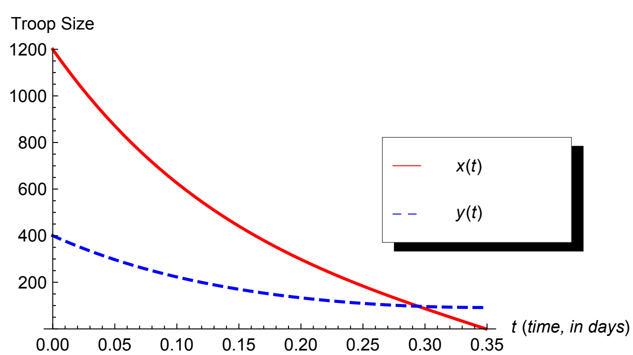
This appendix includes the *Mathematica* workbook code used to generate the solutions and plots of the various model solutions discussed within this report.

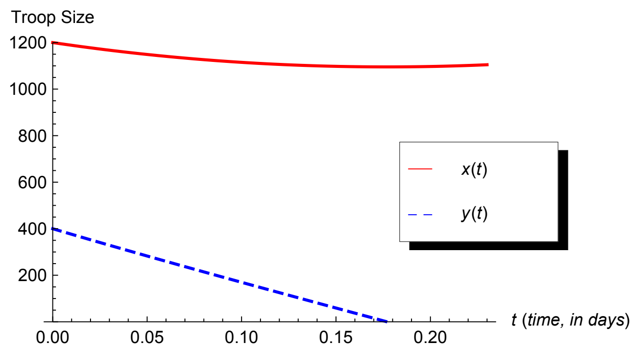
Index

Aimed Fire

```
Needs ["PlotLegends`"]
c = 3;
d = 2;
x20 = 250 ;
y20 = 400 ;
tfinal2 = 0.23 ;

soln = DSolve [
    {x' [t ] == -c * y[t ], y' [t ] == -d * x[t ], x[0] == x20 , y[0] == y20 }, {x, y}, {t , 0, tfinal2 }];
A2 = Plot [{x[t ]} /. soln , {t , 0, tfinal2 }, PlotRange → {0, 1200 }, PlotStyle → {Red }];
B2 = Plot [{y[t ]} /. soln , {t , 0, tfinal2 }, PlotRange → {0, 1200 }, PlotStyle → {Blue }];

Show [Plot [{x[t ] /. soln , y[t ] /. soln }, {t , 0, tfinal2 },
  PlotRange → {0, 400 }, PlotStyle → {Red , {Blue , Dashed }},
  PlotLegend → {Style ["x(t )", Italic ], Style ["y(t )", Italic ]},
  AxesLabel -> {Style ["t (time , in days )", Italic ], "Troop Size "}]]
```

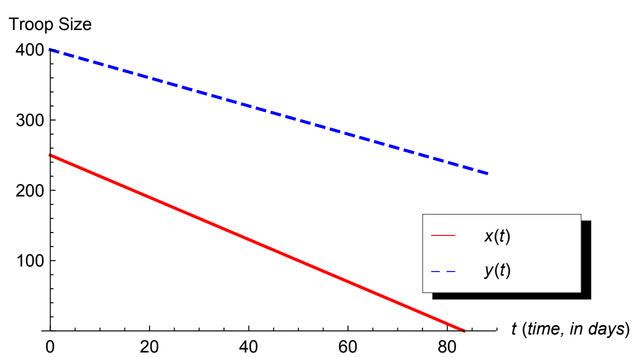


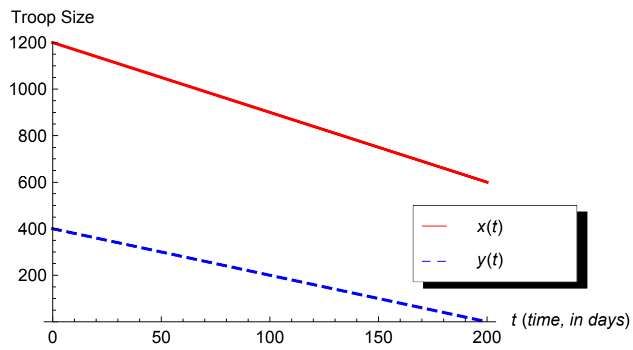


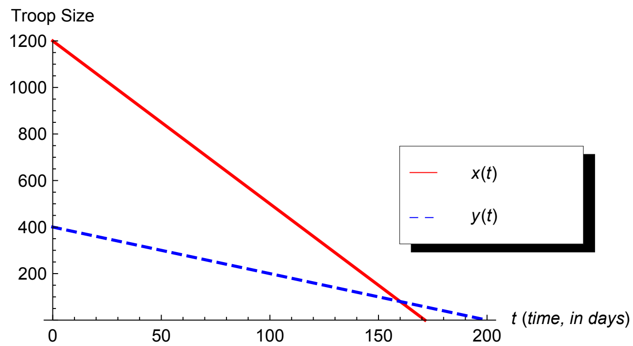


Ancient Warfare

```
Needs ["PlotLegends`   "]
e = 3;
f = 2;
x30 = 250 ;
y30 = 400 ;
tfinal3 = 88 ;

s2 = DSolve [{x ' [t ] == -e, y ' [t ] == -f , x[0] == x30 , y[0] == y30 }, {x, y}, {t , 0, tfinal3 }];
A3 = Plot [{x[t ]} /. s2 , {t , 0, tfinal3 }, PlotRange → {0, 1200 }, PlotStyle → {Red }];
B3 = Plot [{y[t ]} /. s2 , {t , 0, tfinal3 }, PlotRange → {0, 1200 }, PlotStyle → {Blue }];

Show [A3, B3];
Show [Plot [{x[t ] /. s2 , y[t ] /. s2 }, {t , 0, tfinal3 },
   PlotRange → {0, 400 }, PlotStyle → {Red , {Blue , Dashed }},
   PlotLegend → {Style ["x(t )", Italic ], Style ["y(t )", Italic ]},
   AxesLabel -> {Style ["t (time , in days )", Italic ], "Troop Size "}]]
```



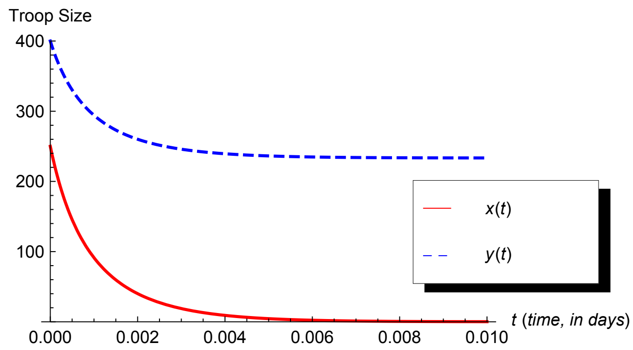


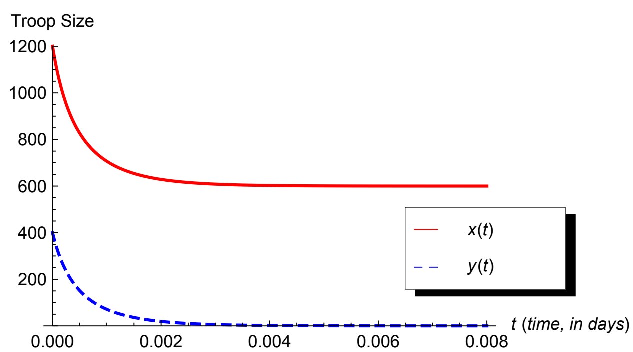


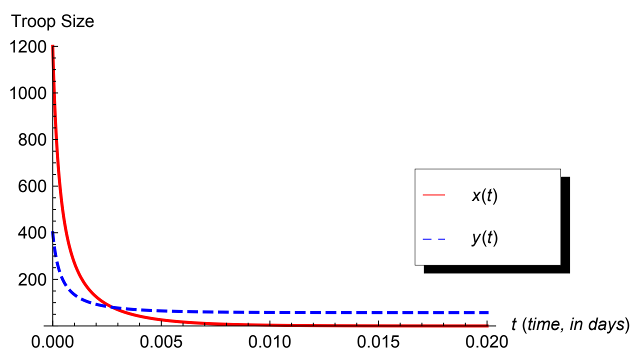
Area Warfare:

```
Needs ["PlotLegends`"]
a = 3;
b = 2;
x0 = 250 ;
y0 = 400 ;
tfinal = 0.010 ;

s = DSolve [{x ' [t ] == − a ∗ x [t ] ∗ y [t ], y ' [t ] == − b ∗ x [t ] ∗ y [t ], x [0] == x0 , y [0] == y0 },
    {x, y}, {t , 0, tfinal }];

A = Plot [{x [t ]} /. s, {t , 0, tfinal }, PlotRange → {0, 400 }, PlotStyle → {Red }];
B = Plot [{y [t ]} /. s, {t , 0, tfinal }, PlotRange → {0, 400 }, PlotStyle → {Blue }];

Show [A, B];
Show [Plot [{x [t ] /. s, y [t ] /. s}, {t , 0, tfinal },
   PlotRange → {0, 400 }, PlotStyle → {Red , {Blue , Dashed }},
   PlotLegend → {Style [" x (t )", Italic ], Style [" y (t )", Italic ]},
   AxesLabel → {Style [" t (time , in days )", Italic ], " Troop Size " }]]
```

Solve ::ifun : Inverse functions are being used by Solve , so
some solutions may not be found ; use Reduce for complete solution information . ≫





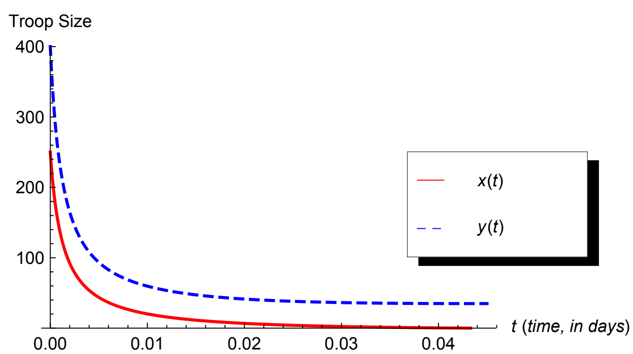


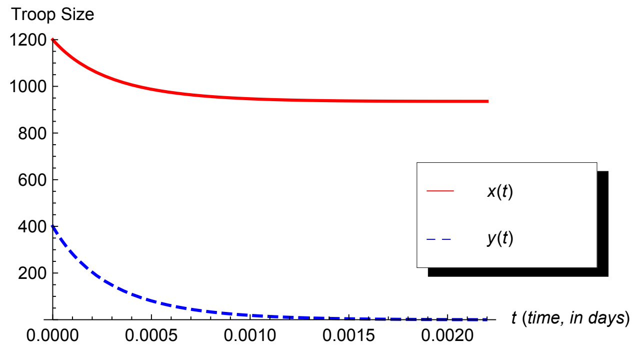
Modern Warfare + Area Fire

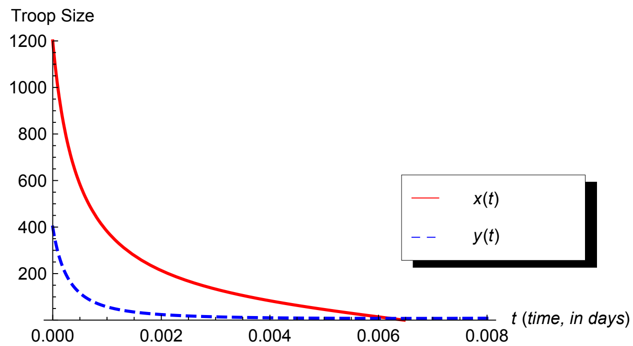
```
g = 3;
h = 2;
i = 2;
j = 3;
x40 = 250 ;
y40 = 400 ;
tfinal4 = 0.045 ;

s3 = NDSolve [{x ' [t ] == -g * y [t ] - h * x [t ] * y [t ],
    y ' [t ] == -i * x [t ] - j * x [t ] * y [t ], x [0] == x40 , y [0] == y40 }, {x, y}, {t , 0, tfinal4 }]
A4 = Plot [{x [t ]} /. s3 , {t , 0, tfinal4 }, PlotRange → {0, 400 }, PlotStyle → {Red }];
B4 = Plot [{y [t ]} /. s3 , {t , 0, tfinal4 }, PlotRange → {0, 400 }, PlotStyle → {Blue }];

Show [A4, B4 ];
Show [Plot [{x [t ] /. s3 , y [t ] /. s3 }, {t , 0, tfinal4 },
  PlotRange → {0, 400 }, PlotStyle → {Red , {Blue , Dashed }},
  PlotLegend → {Style [" x (t )" , Italic ], Style [" y (t )" , Italic ]},
  AxesLabel → {Style [" t (time , in days )" , Italic ], " Troop Size " }]]
```

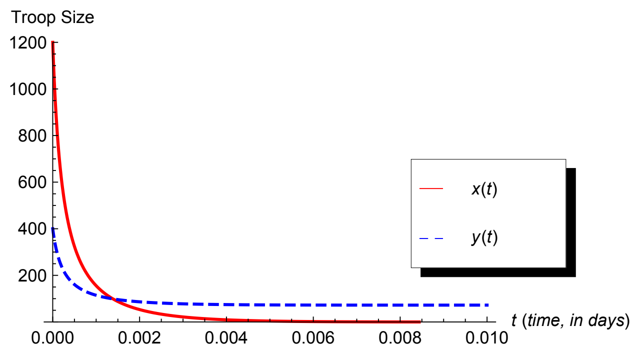
{{x → InterpolatingFunction [Domain : {{0. , 0.045 }} Output : scalar],

y → InterpolatingFunction [Domain : {{0. , 0.045 }} Output : scalar]}}









Simplify [((250 − 200 * Sqrt [6]) * ⅇ ^ {Sqrt [6] * t } + (250 + 200 * Sqrt [6]) * ⅇ ^ {−Sqrt [6] * t }) / 2]

$$\left\{-25\ e^{-\sqrt{6}\ t}\left(-5-4\sqrt{6}+\left(-5+4\sqrt{6}\right)e^{2\sqrt{6}\ t}\right)\right\}$$