Infinite Snowman

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Abstract
In this research paper we explore how to create an infinite snowman within an isosceles triangle to find the sum of its total area. Our main objective was to find the sum of the areas of the circles inscribed within the isosceles triangle.

1 Introduction

Welcome to Winter Wonderland! The place where everyone can create their own best friend - a snowman! But not just a regular snowman, we are talking about an infinite snowman! Imagine creating a snow-friend that is as tall - or taller - than you. With our machines and experts, we can create a snowman to your own liking. You can customize it with different clothes and accessories. We can even give you explicit details about its build and area at no extra cost! Your snowfriend comes with a lifetime warranty, so you can experience every milestone of your life with them.

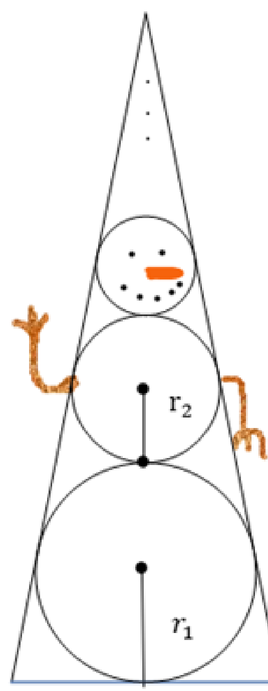

Figure 0: The infinite snowman is created by the first radius, r_1, the second radius, r_2, and so on to find the sum of the total area.

2 Definitions and Development

In the mathematical branch of geometry, we explore the many concepts that pertain to the relationships and properties of points, lines, solids, space, and more. During the history of mathematics, problems with circles inscribed within different *n*-gons, and circles circumscribed about different *n*-gons have been considered. When a circle is inscribed within a shape, then each of the other shape's edges touch the circle once. Thus, we say that each edge is tangent to the circle at a point. Visually, we have a circle within a shape. When circles are circumscribed about a shape, then each side of the shape touches the circle once - they lie on the circle. Visually, it will appear that a polygon is inside a circle.

Although many different kinds of geometries exist, we will be working with Euclidean Geometry and working in 2-space. We will be calculating the areas of circles inscribed in an isosceles triangle by deriving a formula. In order to derive a formula for the area, we must also derive a formula that will determine the measurement of the radii of our circles that create our infinite snowman

When working on paper, we will want to create an isosceles triangle first - a triangle with two sides of equal length. Once we have created our triangle, we create the bottom circle of our snowman that is inscribed within our triangle. This will be the largest circle of our snowman. It must be noted that the body of our snowman cannot have a radius that will bypass the sides of the triangle, it must not be less than the sides of the triangle, and it must be completely tangent to a point (see Figure 1). This indicates that the circles of our snowman will touch the sides of the triangle exactly at a point. We draw an angle bisector from the top angle of our triangle, down to the base denoted *b*. This is now our height denoted *h*.

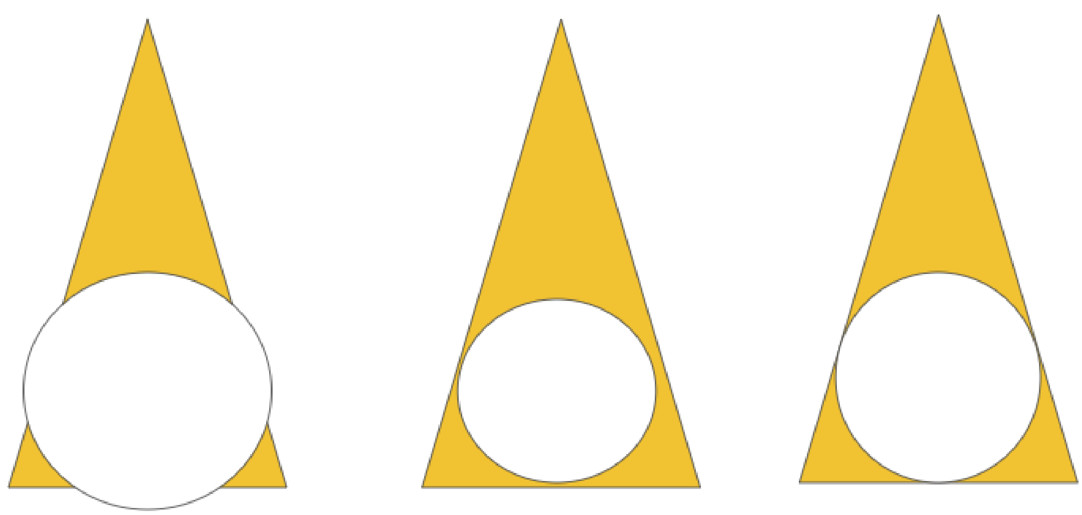

Figure 1: The circle must be tangent to the triangle on three sides.

Focusing only on the triangle on the right side, we note that a right angle has now been created (see Figure 2) because of the perpendicular lines of our height and our base. We now have two triangles.

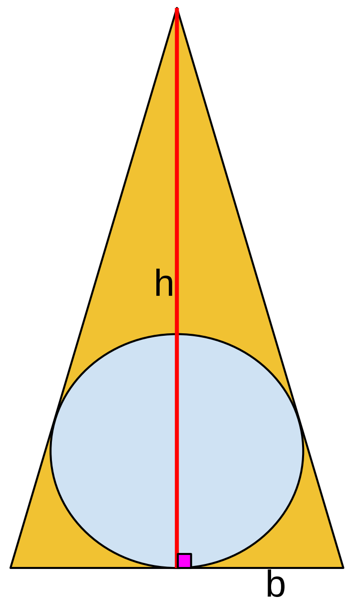

Figure 2: Two right triangles were formed.

We then examine the radius of the inscribed circle, denoted *r*, on our circle from its center down to the base. We note that there are two different approaches that can be taken to find the measurement of our first radius. The first approach is to extend another radius of the circle to the angle at the right side of our triangle to create a right triangle. This will now be the hypotenuse of the right triangle. We flip the triangle at its hypotenuse and create another similar triangle (see Figure 3). We note that the right side of the isosceles triangle now contains *b*, and a new radius is formed.

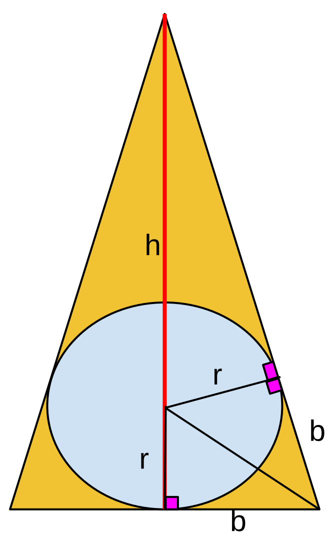

Figure 3: When creating a hypotenuse, another radius is formed and the base is part of the side.

We are able to find our first radius if we use our trigonometric identity of tangent. Because we have two right-angled triangles, we will focus on the one on the right side (see Figure 4).

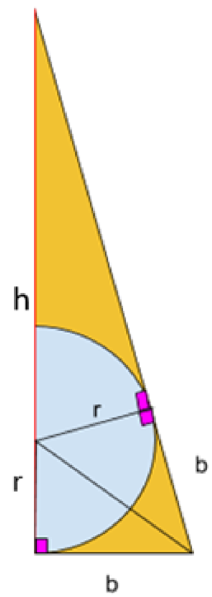

Figure 4: We will use the right-angled triangle on the right when using the trigonometric identity of tangent.

We use this triangle to find the tangent of the angle. First we have to find the size of the angle θ. Notice that there are two right angles within the bigger triangle. Across from each of their right angles is an angle θ (see Figure 5); combining both will give us an angle of 2θ for the bigger triangle.

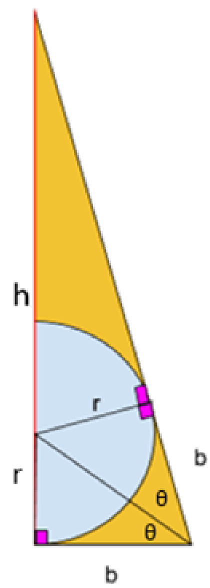

Figure 5: Our angle for the bigger triangle is 2θ because it is composed of the angles of the two smaller right triangles.

The angle of 2θ will allow us to solve for the radius when finding its tangent. Thus

$$tan\ 2\theta\ =\ \frac{h}{b}.$$

Using 3 as the height, *h*, (opposite of the angle) and 2 as the base, *b*, (adjacent to the angle), the tangent ratio is (see Figure 6)

$$tan\ 2\theta\ =\ \frac{3}{2}.$$

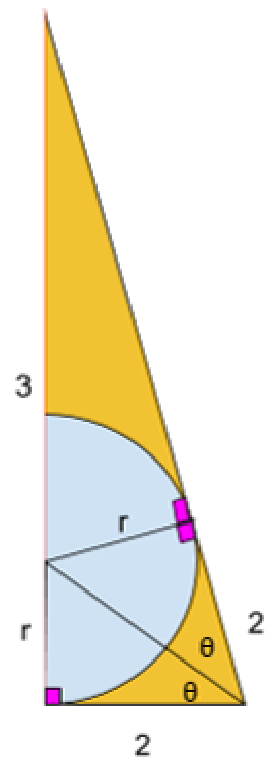

Figure 6: The triangle has a height of 3 and a base of 2.

To get the value of θ, we divide by 2 and take the arctangent, thus

$$\frac{\left(tan\left(\frac{3}{2}\right)\right)^{-1}}{2}\ \approx\ 28.15\ degrees.$$

The angle θ is approximately $28.15\ degrees$ and we can now use it to find the radius by using the trigonometric function of tangent. We include the computations below.

$$tan\ \theta\ =\ \frac{r}{b}$$

$$tan\ (28.15)\ \approx\ \frac{r}{2}$$

$$2\ (tan\ (28.15))\ \approx\ r$$

$$r\ \approx\ 1.07$$

We would have to continue this process for the rest of our radii, but instead, let's create a formula for the radii of our snowman's body to find its total area. We will use this method as a resource tool to check that our radius formula is accurate.

The second approach that we can take incorporates the steps we have taken stated above. But here, we are able to have another triangle with an alternating exterior angle that corresponds to the bigger triangle (see Figure 7).

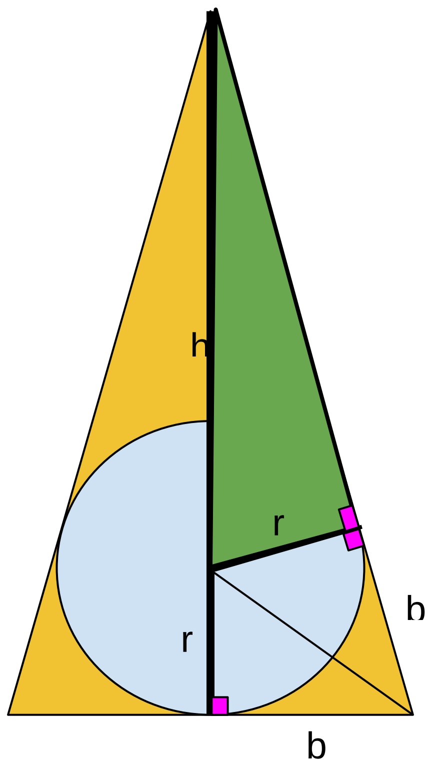

Figure 7: A new triangle is created (in green), which is similar to the larger triangle.

We note that we can also use the right triangle created when it was flipped across the hypotenuse and have supplementary angles. Once we visually observe the steps that we have taken, we see that a radius acts as the base for this new triangle. To obtain the value of the radius, we can use the Pythagorean Theorem on the new triangle on top. One leg of the right triangle will be the side where the right angle is, which will be

$$\sqrt{b^2 + h^2} - b.$$

At the end we have a (– *b)* because we subtract the length of the triangle that was flipped, which was the base, to obtain the side of the smaller top triangle. The other leg is the radius that was created when the triangle was flipped because it is now the base. The hypotenuse will be

$$h - r$$

because there are two radii from our circle that we need to subtract from the height to obtain the scaling factor for the smaller triangle (see Figure 8 for full image).

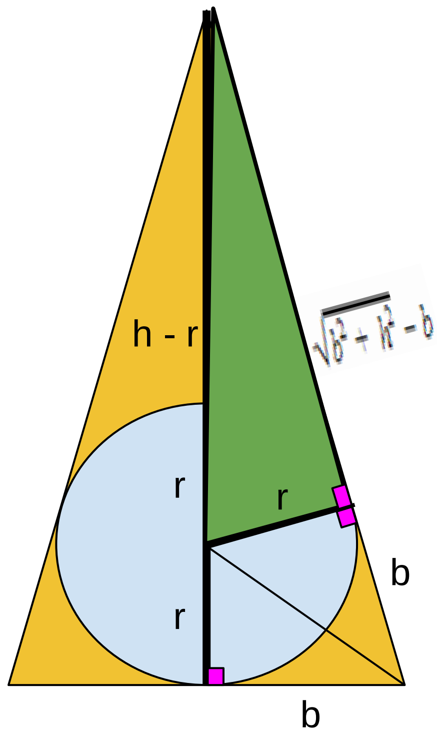

Figure 8: We use the Pythagorean Theorem to find the sides and hypotenuse of the green triangle.

Using the Pythagorean Theorem, we can calculate our radius by generating a formula. Because we need the exact values of the radii of the infinite snowman, this formula is beneficial to finding the sum of the total area of the infinite snowman. The formula we obtain for the radius is

$$r = \frac{b}{h}\left(\sqrt{b^2 + h^2} - b\right)$$

and is proven on page 8. When we use actual numbers in our formula, we must obtain the same value for the radius that we had when we computed our trigonometric identity of tangent. It is important to use the formula because it will give us exact values that will create our circles to be tangent to the sides of the triangle. We use the radius formula to determine the value of our radius and observe that we obtain the same value as we did when we used the trigonometric identity of tangent.

$$r = \frac{2}{3}\left(\sqrt{2^2 + 3^2} - 2\right) \approx 1.07$$

As previously mentioned, we will also need a scaling factor. To find the scaling factor for our triangle, we must concentrate on the heights of two similar triangles. The first triangle will be our original triangle and if we draw a tangent line - parallel to the base - on top of our first circle, we will have another smaller, similar triangle (see Figure 9). Observing that our larger triangle has height *h*, we see that our second triangle has a height of

$$h - 2r,$$

where the *2r* is the two radii that our first circle has. Noticing this height difference, we determine that the scaling factor is

$$f = \frac{h - 2r}{h} = 1 - 2\frac{r}{h}.$$

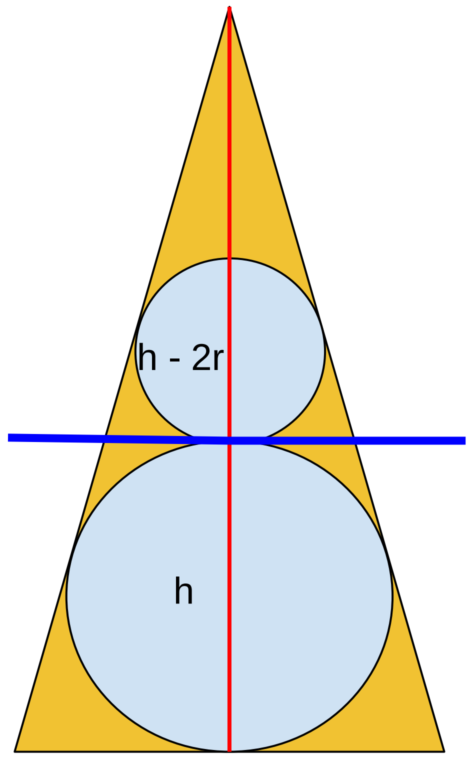

Figure 9: We use the heights of the smaller and larger triangles to create a scaling factor.

3 Results

With the Pythagorean Theorem, we obtain a formula to determine every radii in our snowman. We begin with the appropriate values to obtain

$$\left(\sqrt{b^2 + h^2} - b\right)^2 + (r)^2 = (h - r)^2.$$

We compute and simplify to generate the formula for our radius.

$$2b^2 + h^2 - 2b\sqrt{b^2 + h^2} + r = h^2 - 2hr + r^2$$

$$2hr = 2b\sqrt{b^2 + h^2} - b^2$$

$$hr = b\sqrt{b^2 + h^2} - b^2$$

Our formula becomes

$$r = \frac{b}{h}\left(\sqrt{b^2 + h^2} - b\right).$$

With the calculated radius, we are able to create the first circle of the snowman using the first radius, r_1, and create the second circle using the second radius, r_2, and we continue this pattern for the rest of the snowman's body (see Figure 10).

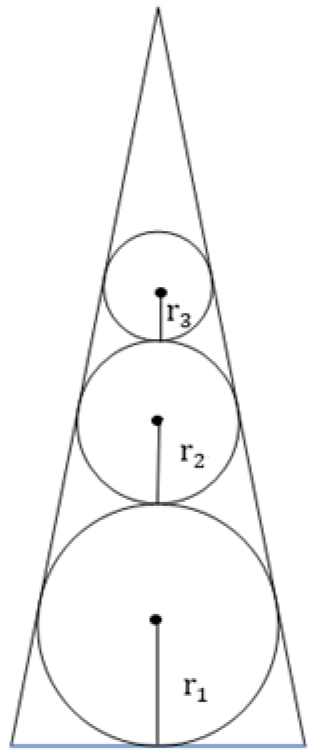

Figure 10: We add the areas of the circles to obtain the total sum.

To find the sum of the total area of our snowman, we must add up every area of each circle.

$$A = \pi r_1^2 + \pi r_2^2 + \pi r_3^2 + \ldots$$

We must multiply by our scaling factor because the body of our snowman becomes smaller every time.

$$= \pi r^2 + \pi(fr)^2 + \pi(f^2 r)^2 + \ldots$$

We continue with our computations to generate a formula for the sum of the total area of our infinite snowman.

$$A = \pi r^2 \left(1 + f^2 + (f^2)^2 + (f^2)^3 + \ldots\right)$$

$$A = \pi r^2 \left(\frac{1}{1 - f^2}\right)$$

We substitute the scaling factor with the formula found

$$A = \pi \frac{b}{h}\left(\sqrt{b^2 + h^2} - b\right)^2 \left(\frac{1}{1 - \left(\frac{h - 2r}{h}\right)^2}\right),$$

as well as the radius. The formula for the sum of the total area is

$$A = \pi \frac{b}{h}\left(\sqrt{b^2 + h^2} - b\right)^2 \left(\frac{1}{1 - \left(\frac{h - 2\frac{b}{h}\left(\sqrt{b^2 + h^2} - b\right)}{h}\right)^2}\right).$$

As mentioned before, we worked with a base of 2 and a height of 3. We can determine the exact sum of the total areas of the infinite snowman when we substitute these values into the area formula and obtain the total area to be

$$A \approx 3.30.$$

4 Conclusion and Directions for Further Research

Different mathematicians created a foundation for a case where three circles are inscribed within an equilateral triangle, the mathematician that is most known for this case is Giovanni Francesco Malfatti (1731-1807). Malfatti wanted to find the optimal volume of three different spheres carved into a triangular prism. There exists a 2-space layout that indicates Malfatti's thought process. The way the circles were arranged inside the triangle altered the area. Initially, Malfatti arranged the circles to be identical in size, while simultaneously being tangent to two sides and two other circles (see Figure 10).

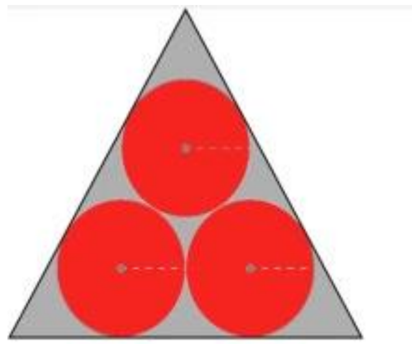

Figure 10: The 2-space model of Malfatti's initial thought on how three circles should be arranged to have an optimal area. In 3-space, Malfatti was looking for an optimal volume.

It was later revealed that there exists another arrangement in where the greater area could be given when a big circle is tangent to all sides of the triangle, and then two smaller triangles being tangent to the big circle, a side length, and the base of the triangle. (see Figure 11).

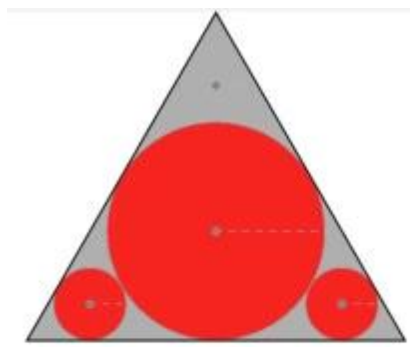

Figure 11: The three circles arranged to have an optimal area.

A Japanese mathematician by the name of Ajima Naonobu studied the point in the triangle created inside of the tangent circles - which is known as the Ajima-Malfatti Point (see Figure 12).

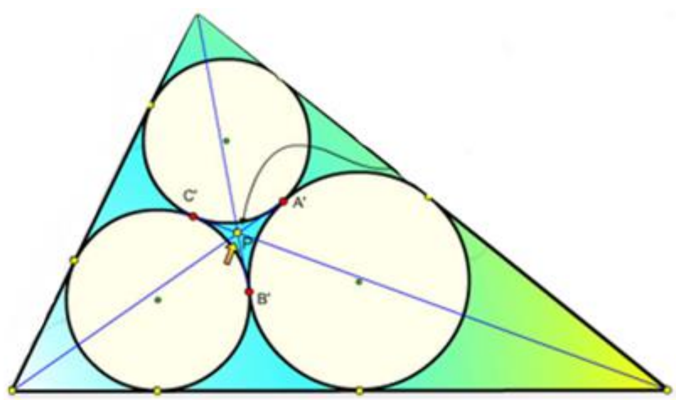

Figure 12: The Ajima-Malfatti Point created by tangent circles within a triangle.

As we have observed, there are various ways to work with circles inscribed in different polygons. For further research, we can find the volume for an infinite amount of spheres in a triangular prism. Along with this, we can look further into the two different approaches that were discovered. To take this research further, it can also be possible to look into triangulation of different *n*-gons with infinite amount of circles inside each triangle (see Figure 13).

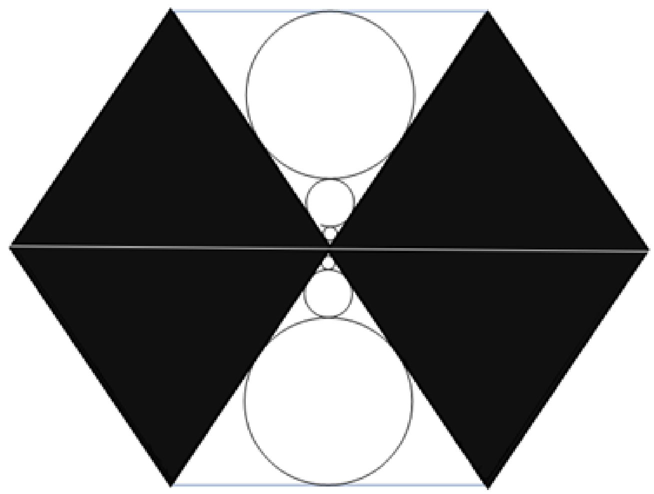

Figure 13: Triangulation of a hexagon with an infinite snowman inscribed in the triangles.

References

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