Mathematically Modeling the Josephson Junction

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Abstract

Josephson junctions are widely used in applications where a very precise voltage source is required, one example being metrology, where they are used to help define the values of several fundamental constants of nature. Modeling the Josephson junction furthers our understanding of the behavior of the junction which has had profound impacts in the metrological world. This junction and its voltage can be described by a nonlinear differential equation similar to that of the simple pendulum. Differential equations of this type are difficult to understand and very difficult if not impossible to solve explicitly. After deriving this nonlinear differential equation using quantum mechanical arguments, the solutions are found independently using both analytical and numerical approaches. The qualitative behaviour of solutions change at a critical current, resulting in either no voltage across the junction or voltage asymptotically approaching the classical Ohm's law. Further modeling of this system could include modeling specific behaviors including the AC and DC Josephson effects.

1 Introduction

Mathematical modeling allows us to look at a problem trivial or not, and capture the behavior of a given system by writing down an equation or a system of equations. These equations provide valuable qualitative and quantitative information for the system in question. To extract this information, these equations are manipulated and analyzed either numerically or analytically.

This method will be used to help us understand the behaviour of Josephson junctions, but before deriving the model, it is useful to discuss some of the underlying physics behind these junctions as well as their basic construction.

Definition 1. Josephson junctions are components in electrical circuits which are constructed by separating two superconductive materials by a non-superconductive material of at most a few atoms thick. A basic junction can be seen in Figure 1.

This non-superconductive region can be made of any material that is not superconducting provided that the material is not too thick. The exact thickness depends on the material used and is related to the quantum nature of the junction, but for reference it can range from a few atoms to a few microns thick. For the purposes of this thesis, we are not concerned with the thickness of this region as we are focusing on the point Josephson junction.

The first important physics topic that governs the behaviour of the Josephson junction is superconductivity. Superconductivity is the result of a behavior change in the electrons in a given material. This change happens at temperatures near absolute zero for most materials but can be higher for some high temperature superconductors. The exact transition point from normal conductivity to superconductivity is known at the critical temperature and it is at this point that the elections change their behavior.

When a material is in a superconductive state, all of its electrons are in the same energy state and form Cooper pairs. These Cooper pairs are two weakly coupled electrons which are treated as one particle. Since all of these Cooper pairs are in the same energy state, they can be described by one wave equation which will be used to derive the model of the Josephson junction. The transition to superconductivity also changes the type of current that exists within the material. The current within a superconductor is called supercurrent, which behaves differently and is therefore described by different equations then traditional current. This discrepancy is critical in modeling the Josephson junction as it operates with supercurrent, not regular current, though it is integrated within traditional current circuits. To capture the behaviour of the Cooper pairs and correctly model the supercurrent in the junction, we will need some quantum mechanics and parts of Schrodinger's wave equation.

The quantum mechanics that we need arises from how the suppercurrent manifests in the Josephson junction. Recall that the junction is created by the separation of two superconducting regions by a very small non-superconducting region. The supercurrent is created by the Cooper pairs quantum tunneling between the two superconducting regions across the non-superconducting region. This tunneling is due to the wave like properties of electrons which act like probability densities for the positions of the electrons, and are described by Schrodinger's equation. Essentially, the equation gives the probability of a particle being in any given region. Some regions have potential barriers, like the non-superconducting region in the Josephson junction, that decrease the probability of a particle existing on the other side of the potential barrier. This is why the non-superconducting region has a maximum thickness. If the region is too large, the Cooper pairs would not be able to tunnel across.

The supercurrent created by quantum tunneling Cooper pairs in the Josephson junction is unique in the sense that the current is created without a voltage developing across the non-superconducting region. But,when a current is applied past a threshold unique to each junction, a well-defined voltage will develop.

This well-defined voltage is the reason that Josephson junctions are used in precision applications such as metrology the study of measurement. Every

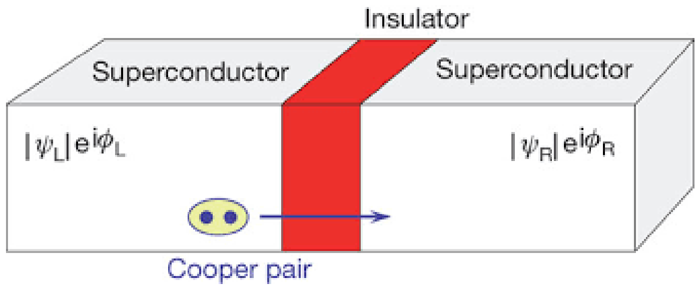

Figure 1: This is the basic construction of a Josephson junction with two superconducting regions separated by a non-superconducting region. This figure also depicts the Cooper pair tunneling across the non-superconducting region which creates the unique supercurrent without a voltage. The expressions in each superconducting region represent the probability density and phase of the cooper pairs in each respective region. These equations are explained more fully in Section 2. This figure was obtained from [3].

measurement of mass and current made in our daily lives can be traced back to its respective SI base unit which the Josephson junction was involved in defining. Since this electrical component is intimately involved in our measurement of the world, it is important that we understand its behaviour.

2 Definitions and Development

To accurately model the Josephson junction and capture its behaviour, we will start with quantum mechanical arguments to describe the supercurrent created by the tunneling Cooper pairs. These arguments lead us to the Josephson current and voltage relations. These equations are fundamental to the behaviour of the junction and are used to derive the second order non linear non homogeneous differential equation which describes the current of the junction as a function of its voltage. Equations of this type are very hard to solve and their closed-form solutions are difficult to find, therefore we will be using a number of simplification techniques,the most notable of which being nondimensionlilzation. This simplification will be used in both the analytical approach as well as the numerical approach in Mathematica. Although it is possible to find a closed-form solution using elliptic integrals of the first kind, I will be focusing on further simplifying the equation and the numerical approach to find how voltage depends on current in the Josephson junction.

2.1 Physics Principles

To begin the analysis of the Josephson junction, we need to capture the behaviour of the electrons in the two superconducting regions. This will be done using the framework of quantum mechanics and a form of Schrödinger's equation.

Since the two regions are superconducting, all of the electrons on a given side are described by one wave function. The amplitude for an electron on the left region will be ψ_1 and similarly for the right side, the amplitude will be ψ_2. These two amplitudes are coupled by the non-superconducting junction in such a way that yields the following two equations

$$i\hbar\frac{\partial\psi_1}{\partial t} = E_1\psi_1 + K\psi_2 \tag{1}$$

and

$$i\hbar\frac{\partial\psi_2}{\partial t} = E_2\psi_2 + K\psi_1. \tag{2}$$

Both of these coupled equations have the same K parameter. This term is the oscillation amplitude between the junction and it is specific to each junction based on the specific materials the superconductor is made out of as well as the thickness of the junction. On the left hand side of the equations, $i = \sqrt{-1}$ and $\hbar$ is Plank's constant divided by 2π. The parameters E_1 and E_2 represent the energy of the system.

These equations are as far as the derivation can go with the isolated Josephson junction, so to further the analysis, the junction is integrated in a simple circuit. This is achieved by attaching a battery to the junction such that one superconducting region is connected to the positive terminal of the battery and the other superconducting region is connected to the negative terminal. This creates a voltage difference between the two superconducting regions such that

$$E_1 - E_2 = qV. \tag{3}$$

To satisfy this equation, we let $E_1 = \frac{qV}{2}$ and for $E_2\frac{-qV}{2}$. These values for energy are then substituted into Equations 1 and 2 yielding,

$$i\hbar\frac{\partial\psi_1}{\partial t} = \frac{qV}{2}\psi_1 + K\psi_2 \tag{4}$$

and

$$i\hbar\frac{\partial\psi_2}{\partial t} = \frac{-qV}{2}\psi_2 + K\psi_1. \tag{5}$$

At this point it is useful to consider the forms that the amplitudes of ψ_1 and ψ_2 can take. Our wave functions ψ_1 and ψ_2 are the the wave functions for any given Cooper pair in their given superconducting region. The study of superconductivity tells us that the modulus of a wave function in this case will be proportional to the square root of the electron charge density in the superconductor, denoted ρ. This allows us to set the wave functions equal to the charge density multiplied by a phase term. The phase term essentially shifts the wave depending on the value of the exponent. This allows us to set the wave functions

$$\psi_1 = \sqrt{\rho_1} e^{i\phi_1} \tag{6}$$

and

$$\psi_2 = \sqrt{\rho_2} e^{i\phi_2}. \tag{7}$$

Here, ρ_1 and ρ_2 are the electron charge densities at given points in each respective superconductive region and ϕ_1 and ϕ_2 are the phases for each region. These wave functions are then substituted back into the coupled Equations 4 and 5. Though it isn't shown, the electron charge densities and phases for both regions are functions of time. After differentiating and simplifying, the real and imaginary parts are separated. This results in four equations, one for each of the time derivatives of phase and charge density which are

$$\dot{\rho_1} = \frac{2K}{\hbar} \sqrt{\rho_1 \rho_2} \sin\phi, \tag{8}$$

$$\dot{\rho_2} = \frac{-2K}{\hbar} \sqrt{\rho_1 \rho_2} \sin\phi, \tag{9}$$

$$\dot{\phi_1} = \frac{-K}{\hbar} \sqrt{\frac{\rho_2}{\rho_1}} \cos\phi - \frac{qV}{2\hbar}, \tag{10}$$

$$\dot{\phi_2} = \frac{-K}{\hbar} \sqrt{\frac{\rho_1}{\rho_2}} \cos\phi + \frac{qV}{2\hbar}. \tag{11}$$

For simplicity, $\phi_2 - \phi_1$ is written as ϕ. The change in charge densities ρ_1 and ρ_2 are analogous to current since current is a measure of how the density of charge is changing at a given point. Looking at Equations 8 and 9, we can see that $\rho_1 = -\rho_2$ and since we derived this by integrating the Josephson junction into a simple circuit, the charge densities are moving from one region to the other. As one increases, the other decreases by the same amount. This allows us to arbitrarily let $\rho_1 = I$ or the bias current through the junction.

Definition 2. The **bias current** denoted I, is defined as the supercurrent that passes through the junction by quantum tunneling Cooper pairs.

Since there is a constant influx of charge density from the battery, the charge densities ρ_1 and ρ_2 essentially stay constant and equal to the standard superconductor electron density ρ_0. One final simplification is to lump the remaining constants into a term called the critical current, $I_c = \frac{2K\rho_0}{\hbar}$.

Definition 3. The **critical current** denoted I_c, is equal to $\frac{2K\rho_0}{\hbar}$ which is a value specific to each junction that acts as a threshold between qualitatively different behaviours in the junction.

2.2 Differential Equations Model

Overall these simplifications yield what is known as the Josephson current-phase relation

$$I = I_c \sin \phi. \tag{12}$$

Before discussing and analyzing the Josephson current-phase relation, we will focus on Equations 10 and 11. Since the Josephson current-phase relation is in terms of ϕ or $\phi_2 - \phi_1$, we will start by subtracting their derivatives to find $\dot{\phi}$. The first term drops out leaving us with

$$\dot{\phi} = \dot{\phi}_2 - \dot{\phi}_1 = \frac{qV}{\hbar}.$$

This more useful when solved for voltage and when q is substituted. It is important to remember that in superconductivity, because of the Cooper pair formation, the charge q needs to account for two electrons meaning $q = 2e$ yielding,

$$V = \frac{\hbar \dot{\phi}}{2e}. \tag{13}$$

This equation is known as the Josephson voltage-phase equation. It describes the voltage dependence on the change in phase difference between the two superconducting regions. The Josephson current-phase relation 12 and this voltage equation 13 provide the ability to examine the current and voltage dependence on the difference in phase between the two superconducting regions. Ultimately, we wish to examine the voltage dependence on the bias current, so further derivation is required, but first it is useful to discuss the behaviour of these equations and how they manifest in the Josephson junction.

It is also important to note that Equations 12 and 13 can be manipulated further to accommodate a more complicated Josephson junction system, for example one that includes the presence of an external magnetic field. But, for our simple symmetric junction, the equations in their current form sufficiently model its behaviour.

Equation 12 is one of the two equations that describe the behavior of the Josephson junction, where ϕ is the phase of the Cooper pairs in each superconducting region, I is the bias current of the circuit and I_c is the critical current of the system. Equation 12 tells us that the bias current in the Josephson junction is sinusoidal and depends on the difference in phase between the two superconducting regions. As this difference increases, the bias current approaches the value of the critical current at which point this equation will no longer provide an accurate model for the bias current through the Josephson junction.

The Josephson current-phase relation is only true when the bias current is less than the critical current. This can be seen by looking at the values that $\sin(\phi)$ can take. Since ϕ is a real number, $-1 < \sin(\phi) < 1$, which reduces the value of the critical current. The bias current can exceed the value of the critical current, but when the critical current is greater, this equation holds.

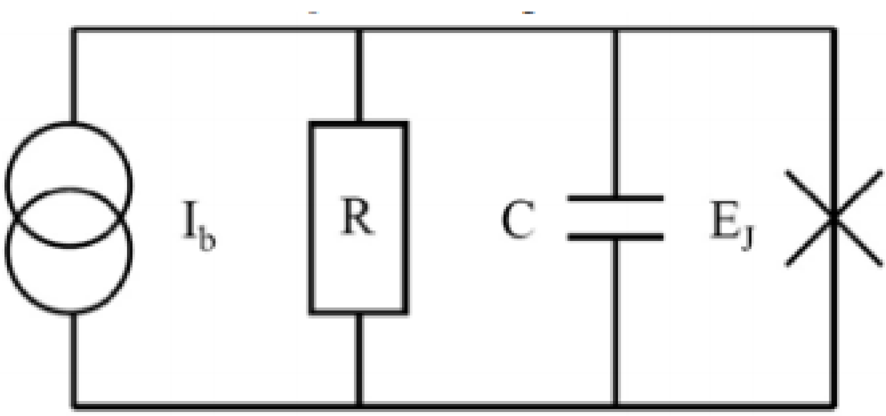

Figure 2: In the circuit, the Josephson junction is represented by the X, the resistor by the rectangle, the capacitor by the separated parallel lines and the current applied to the circuit is represented by the intersecting circles. In this circuit, both regular current and tunneling current can travel and to model this, it is use full to analyze this circuit using Kirchhoff's voltage and current laws. This figure was obtained from [3].

It is important to note that Equation 12 requires the phase difference between the two superconducting regions to be constant. This produces interesting results in the Josephson voltage-phase relation 13. Since Equation 13 requires the phase difference to be changing, a constant phase difference results in zero potential difference in voltage across the junction despite the nonzero current. This seems to contradict Ohm's law, $V = IR$ but this phenomenon is a result of the quantum nature of the junction and its superconducting regions with zero resistance where Ohm's law is strictly classical.

When the bias current is greater than the critical current, a voltage does form across the junction and that voltage is governed by Equation 13. This is a result of Equation 12 no longer being a true representation of the value of the bias current and the change in phase no longer being constant. The voltage that forms across the junction is directly related to the change in phase difference between the two superconducting regions. The greater the change in phase between the two regions, the greater the voltage that develops across the junction. The problem with the two governing equations for the Josephson junction is that depending on the value of the bias current relative to the critical current, we can either know the current that is passing through the junction or the voltage that is developed across the junction, but not both at the same time. To find an accurate representation of the current when $I > I_c$ and there is a nonzero voltage, the Josephson junction needs to be integrated into a more complicated circuit then our current model.

This new circuit will include two other common circuit elements, namely a resistor and a capacitor in parallel to the Josephson junction. This new circuit is shown in Figure 2 and the analysis of the Josephson junction in this circuit requires the use of Kirchhoff's laws of voltage and current to derive the equation that governs the current when $I > I_c$.

Kirchhoff's voltage law states the sum of the voltages in a closed loop must be zero and his current law states that the sum of the currents entering a node is equal to the sum of the currents exiting a node. More specifically, for the voltage law, the voltage drop across any of the parallel paths in the circuit must be the same. For our circuit, we have three parallel paths so each path must have the same voltage drop. This makes the voltage across each of the three paths the same V. Since we know the voltage across the Josephson junction is the voltage-phase equation 13, we can let the voltage across the resistor and capacitor be described by Equation 13. Kirchoff's current law for our circuit means that the sum of the currents in each branch must be the total current in the circuit, or in our case the bias current. The current through the resistor and the capacitor respectively are $\frac{V}{R}$ where R is resistance and $C\frac{dV}{dt}$ where C is capacitance and the current though the Josephson junction is still $I_c \sin(\phi)$. Setting the sum of these individual currents equal to the bias current yields

$$I = C\frac{dV}{dt} + \frac{V}{R} + I_c \sin(\phi).$$

Since we know by Kirchoff's voltage law that all of the voltages are equal to the voltage-phase equation 13, once we substitute it in for each voltage term in this current equation, we find the equation describing the bias current across the Josephson junction when $I > I_c$ to be

$$I = \frac{\hbar C}{2e}\frac{d^2\phi}{dt^2} + \frac{\hbar}{2eR}\frac{d\phi}{dt} + I_c \sin(\phi). \tag{14}$$

Here, $2e$ is the fundamental charge of two electrons because of the Cooper pairs, $\hbar$ is Planks constant divided by 2π, C is the capacitance of the capacitor and R is the resistance of the resistor. The current when $I > I_c$ is much more complicated than 12 since this is a second order nonlinear differential equation and will require much more analysis to understand and solve than when $I < I_c$.

Unlike Equation 12 which is only true when $\frac{I}{I_c} < 1$, Equation 14 holds for values of I which exceed that of I_c. This allows us to study the relationship between bias current and voltage when the ratio is greater than one and eliminates the restrictions imposed by Equation 12.

It is not immediately obvious what the first two terms within Equation 14 represent, so it's worth discussing since we will be using this equation for the remainder of the derivation. The $\frac{d\phi}{dt}$ term represents the voltage in the junction, since we know from Equation 13, voltage is proportional to $\frac{d\phi}{dt}$. Following the same reasoning the $\frac{d^2\phi}{dt^2}$ term must be the change in voltage. These terms, though derived from a capacitor and a resistor using Kirchoff's laws, describe the voltage within the Josephson junction and relate those values to the bias current flowing through the junction.

Equation 14 in its current form is unnecessarily complicated for analysis so we will be using nondimensionilization techniques to simplify the analysis of the equation to understand the behaviour of the junction. This will not only remove all of the units but simplify the coefficients in front of the differentials. This method will allow us to analytically find solutions to this nonlinear second order differential equation, which otherwise, would be considerably more difficult.

2.3 Nondimensionalization

When modeling physical systems with equations to describe its behaviour, the resulting equations are often accompanied by numerous physical constants and their respective units. These units can make analyzing the model's behaviour challenging, so to circumvent this problem, we will use a process called nondimensionalization. Essentially, nondimensionalization is just a change of variables with the goal of canceling out most, if not all of the units present in the model. To nondimensionilize Equation 14, we will first look at all of the units present in the model.

Since each term in Equation 14 is added together and equal to the bias current, they must have the same units as the bias current, so each term must have units of Amps. To make the equation dimensionless, we will divide by the critical current I_c since it is a constant for a given Josephson Junction and the bias current is variable. This operation has nondimensionilized the equation, but we will still introduce a new time variable to eliminate many of the constants for simplicity. This new time variable will be a dimensionless time τ, such that $qt = \tau$, where q is a proportionality constant and this relation is used to substitute the change of variables in Equation 14 and results in the following equation

$$\frac{I}{I_c} = \frac{\hbar C q^2}{2eI_c}\frac{d^2\phi}{d\tau^2} + \frac{\hbar q}{2eRI_c}\frac{d\phi}{d\tau} + \sin(\phi). \tag{15}$$

The proportionality constant q is now in both of the terms that still have constants in them. This leaves us with two possible convenient values for q depending on which set of constants we wish to eliminate to simplify the equation.

Choosing a value for the proportionality constant that eliminates all of the constants on the second derivative term results in an equation that yields hysteresis. This is a very interesting problem, but requires much more complicated analysis and is beyond the scope of this thesis. So I will be instead focusing on the other value for q and analyzing that resulting equation.

Eliminating the constants on the voltage term by letting $q = \frac{2eI_cR}{\hbar}$, leaves Equation 15 with

$$\frac{I}{I_c} = Q\frac{d^2\phi}{d\tau^2} + \frac{d\phi}{d\tau} + \sin(\phi). \tag{16}$$

Here, Q represents the remaining constants $\frac{2eI_cR^2C}{\hbar}$ whose dimensions all cancel out leaving a simplified dimensionless equation. The value for q substituted in is known as the McCumber parameter.

In this differential equation, Q is the damping parameter, meaning when $Q \gg 1$, the system is underdamped, when $Q \ll 1$, the system is overdamped and somewhere in between, the system is critically damped. To further simplify Equation 16, we will look at just the overdamped case when $Q \ll 1$, which renders the change in voltage term negligible, resulting in

$$\frac{d\phi}{d\tau} = \frac{I}{I_c} - \sin(\phi). \tag{17}$$

This equation relates the change in phase difference between the two superconducting regions to the ratio of the bias current to the critical current. This relation will provide the foundation for the analysis on how the voltage of the Josephson junction is related to the ratio of the currents flowing across the junction.

When the ratio is greater then one, the change in phase is positive. This has more meaning when this phase is substituted in to Equation 13, the Josephson voltage-phase equation. The other case is when the ratio is less then one. But, it is important to remember that this equation was derived to describe the current when $I > I_c$, meaning Equation 17 is only defined when $I > I_c$.

As mentioned previously, Equation 17 is more meaningful when substituted into Equation 13. This allows us to directly analyze how the voltage of the Josephson junction is related to the bias current when $I > I_c$. But due to the change of variables implemented in the nondimensionilization of Equation 14, it's not as simple as directly substituting Equation 17 for the change in phase difference in Equation 13. To account for this we apply the chain rule for $\frac{d\phi}{dt}$ to get $\frac{d\phi}{d\tau}\frac{d\tau}{dt}$ and by differentiating both sides of $qt = \tau$, we know that $\frac{d\tau}{dt} = q = \frac{2eI_cR}{\hbar}$. This allows us to substitute Equation 13 into Equation 17 and find

$$V = I_c R \left(\frac{I}{I_c} - \sin(\phi) \right). \tag{18}$$

This equation captures the behaviour of the voltage in the Josephson junction when the current is greater then the critical current. This equation is more desirable then Equation 13 because instead of the voltage being related to the change in phase difference between the two superconducting regions, it is related to the ratio of the currents. This allows us to compare the voltage of the junction to the current that is flowing through the junction instead of the change in phase difference which is much harder to measure.

3 Results

Since we have manipulated the model into the desired form, with voltage as a function of the current ratio, we will now look at the solutions to the model. The solutions to Equation 18 will be found in two cases, when the current ratio is less then one, and when the ratio is greater then one. These two cases will describe the voltage across the non-superconducting region in the Josephson junction relative to the number of Cooper pairs tunneling across the region.

3.1 Analytical solution

To find the behaviour of the junction, we need to linearly approximate the change in phase difference, Equation 17 since it is oscillatory. Ultimately, this approximation allows us to express the voltage only in terms of the bias current and the critical current instead of the phase difference, since again, its much harder to measure. First it is important to consider the case when $V = 0$. In doing this we find that

$$\arcsin\left(\frac{I}{I_c}\right) = \phi. \tag{19}$$

Since we are taking the inverse sine of the ratio, and we are only concerned with the real solutions, Equation 19 is only true when the ratio is less then one. In other words, this equation is consistent with the behaviour of Equations 12 and 13 when $I < I_c$. This solution also tells us that the voltage can never be zero when the bias current is greater then the critical current.

An interesting case to note in Equation 18 is when $\phi = 0$. When this happens there is no difference in phase between the two superconducting regions of the Josephson junction since $\phi = \phi_2 - \phi_1$, the equation becomes Ohm's law $V = IR$. When the phases are the same, there is no tunneling of Cooper pairs and the Junction can be described using only classical physics. The behaviour of Equation 18 when $I > I_c$ is periodic, or at least $\dot{\phi}$ is. To linearly approximate the voltage equation and write the equation in a more practical form that is only dependent on the resistance and currents, we need to find the period of the change in ϕ, then take the average value of the function over one period.

Since Equation 17 is 2π periodic, the period of its solutions can be found by

$$T = \int_0^{2\pi} \frac{d\phi}{\frac{I}{I_c} - \sin(\phi)}, \tag{20}$$

which, when solved simplifies to

$$T = \frac{2\pi}{\sqrt{\left(\frac{I}{I_c}\right)^2 - 1}}. \tag{21}$$

Now that we know the period of Equation 17, we can find its average value over that period for the linear approximation. The average value of a function over a given region is found by

$$\overline{F}(x) = \int_a^b \frac{1}{b-a} F(x)dx. \tag{22}$$

Since this average is over one period, the interval we will be integrating over is from 0 to T. We will not be integrating over space but over the time τ since the period is the time it takes for the change in the difference in phase to reach the same value. Substituting this into the average value formula, we find

$$\overline{\frac{d\phi}{d\tau}} = \frac{1}{T}\int_0^T \frac{d\phi}{d\tau}d\tau. \tag{23}$$

This simplifies to

$$\overline{\frac{d\phi}{d\tau}} = \frac{1}{T}\int_0^{2\pi} d\phi, \tag{24}$$

where we have substituted the upper bound T for 2π since we know that equation 17 is 2π periodic. Upon integrating we find that

$$\overline{\frac{d\phi}{d\tau}} = \frac{2\pi}{T}, \tag{25}$$

which, when T is substituted, simplifies to

$$\overline{\frac{d\phi}{d\tau}} = \sqrt{\left(\frac{I}{I_c}\right)^2 - 1}. \tag{26}$$

This is the average value of Equation 17 over one period. This value is then substituted into Equation 18 for $\frac{d\phi}{d\tau} = \frac{I}{I_c} - \sin(\phi)$ to linearize the equation so we can more effectively compare the voltage to the ratio of the currents yielding

$$V = I_c R \sqrt{\left(\frac{I}{I_c}\right)^2 - 1}. \tag{27}$$

This is one of the two solutions to Equation 18, more specifically this is the solution when the ratio of the bias current and the critical current is greater then one. The other solution is when the ratio is less then one and the only possible voltage in that case is zero. These two solutions can then be treated as a piecewise function and graphed for the different values of the ratio because they occupy distinct regions when treated as the domain in the Josephson voltage function, when $I < I_c$ and when $I > I_c$.

$$V = \begin{cases} 0, & I \le I_c \\ I_c R \sqrt{\left(\frac{I}{I_c}\right)^2 - 1}, & I > I_c \end{cases} \tag{28}$$

This piecewise voltage function provides a method of studying how the ratio between the bias and critical currents effects the voltage of a Josephson junction and is not limited to only when the ratio is less then one. This function is graphed in Figure 3 where it is shown that the function asymptotically approaches the classical Ohm's law. This provides

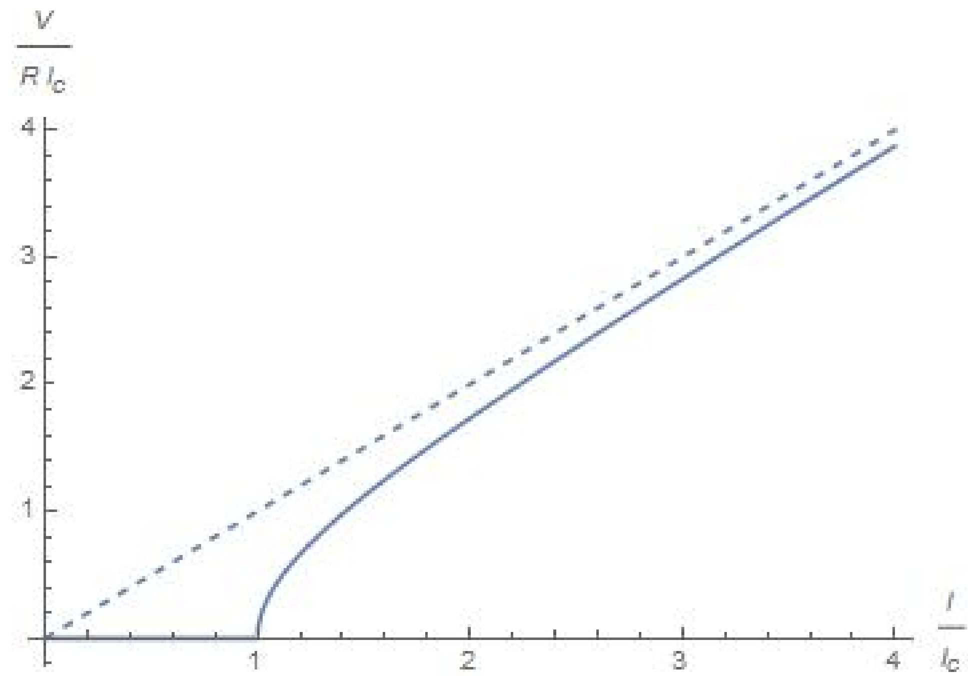

Figure 3: This is the graph of the analytical solution, Equation 28, for the over-damped Josephson junction and clearly shows the relationship between voltage and the current ratio in the junction. As the bias current increases, the voltage across the junction asymptotically approaches the dotted line. This line is the classical Ohm's law $V = IR$, so in the limit as the ratio of bias current to the critical current approaches infinity, the Josephson voltage as a function of the current ratio behaves classically. It is important to note that the y-axis is $\frac{V}{RI_c}$. Since R and I_c are constants, they are essentially just scaling factors and do not qualitatively effect the behaviour of the solution when graphed.

3.2 Numerical Solution

Apart from the analytical approach to understanding the behaviour of the Josephson junction, a numerical approach to solving the equation can be taken and it is equally as insightful. Working in Mathematica, Equation 16 can be numerically evaluated by creating a system of equations that retains the behaviour of the original equation while enabling computational analysis. Since we are solving the system numerically, there is no reason to only consider the overdamped case as we did in the analytic solution.

In order to numerically evaluate the equation in Mathematica, we need to restructure the equation into a form the computer can work with. Computers in general are very proficient in dealing with matrices and since a system of equations is easily converted into a matrix, we will restructure Equation 16 into a system of equations.

This is accomplished by letting a new variable take the place of the first order derivative term, and the derivative of the new variable will take place of the second order derivative term. An added benefit of this substitution is a decrease in order in the differential equation, where Equation 16 was a second order nonlinear differential equation, this new system of equations will only be first order nonlinear differential equations. The new variable will be called λ and will be related to ϕ by the following equations

$$\lambda = \frac{d\phi}{d\tau}\ , \qquad \lambda' = \frac{d^2\phi}{d\tau^2}. \tag{29}$$

Here, $\lambda = \frac{d\phi}{d\tau}$ is proportional to voltage by Equation 13, so $\lambda' = \frac{d^2\phi}{d\tau^2}$ represents a change in voltage. By introducing this new variable λ, we have created a system of equations which retains the original behaviour but is in a form which the computer can understand and manipulate numerically.

The new variables substituted in to Equation 16 results in

$$\lambda' = \frac{I}{QI_c} - \frac{1}{Q}\lambda - \frac{1}{Q}\sin(\phi), \tag{30}$$

where Q still represents the constants left over after nondimensionalization. Both Equation 30 and the first Equation in 29 create the system of equations which Mathematica can numerically solve with a given initial condition. This allows for the behaviour of the system to by analyzed as either the ratio of the critical current and the bias current or by the value of Q.

Using the numerical differential equation solver in Mathematica NDSolve, we found solutions to the system of equations then plotted the solutions on the direction field. This was done both to show the behaviour of the system more clearly as well as how it could vary with different initial conditions and values for the ratio of currents and the voltage. These solutions are found in Figures 4 and 5, where the change in voltage (λ') was plotted on the x-axis and the voltage was plotted(λ) on the y-axis. The solutions depict the behaviour of the bias current as a function of the ratio of currents as well as the voltage that develops across the non-superconducting region.

The solution depicted in Figure 4 represents the junction when the bias current is less then the critical current. In this case, the solution is a fixed point with zero voltage but a nonzero change in voltage (λ'). The change in voltage is nonzero since the junction actively dissipates the charge accumulation which develops as current flows across the junction via tunneling Cooper pairs. This charge is dissipated until until the junction reaches its equilibrium point which is zero voltage and some constant nonzero change in voltage depending on initial conditions.

This solution agrees with the analytic solution in Equation 28 when the bias current is less then the critical current. The numerical solution tells us more about how the system evolves over time in this case. When the system is started at some nonzero initial voltage and change in voltage, in this case the values where $(\lambda', \lambda) = (1, .5)$, the solution approaches the fixed point with no voltage and a constant change in voltage.

The second case of solutions is shown in Figure 5, where we see an oscillating solution. These oscillations coincide with the periodic behaviour predicted by the analytical approach given in Equation 28 and why it was necessary to find the average value over one period. In this solution, the initial condition for (λ', λ) was $(.5, 1)$ and as depicted by the vector field, the Josephson junction has to build up the voltage before the oscillations start.

Both of the numerical solutions verify the analytical solutions with the zero voltage fixed point when the ratio between the bias current and the critical current is less then one and periodic solutions when the ratio is greater then one.

4 Conclusion

The Josephson junction can be solved either analytically or numerically and the solutions verify each other despite the analytical solution being the overdamped junction and the numerical solution being the entire nomdimensionilized junction. This shows that simplifications made in the analytical solution did not qualitatively change the behaviour of the model.

The analytical solutions describe how the voltage of the Josephson junction is related to the ratio of the bias and critical currents. When the ratio is greater than one, no voltage develops across the junction which is also seen in Equations 12 and 13, but they do not describe how the voltage behaves when the ratio is greater the one. This can be seen by looking at the values that $\sin(\phi)$ can take. Since ϕ is a real number, $-1 < \sin(\phi) < 1$, which reduces the value of the critical current. In this case a voltage does develop across the junction and its value asymptotically approaches the value predicted by classical Ohm's law.

The solution in the first case, when the critical current is less then the bias current, represents a spiral sink solution where the fixed point regardless of initial condition is zero voltage with a nonzero change in voltage developing across the non-superconducting gap in the Josephson junction. This is easily seen in the numerical solution shown in Figure 4 where the vector field depicts

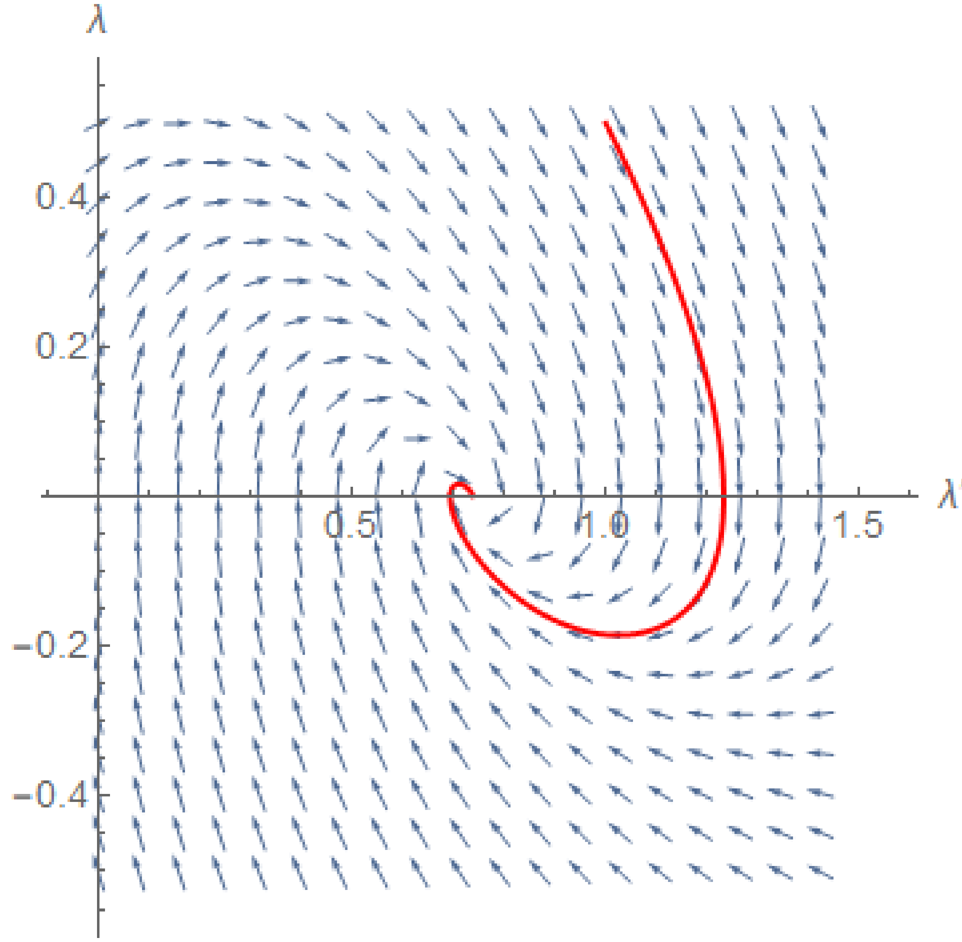

Figure 4: A plot of when the numerical solution when the critical current is grater then the bias current. The solution is plotted on top of the vector field to show the behaviour for different initial conditions. This solution is a spiral sink to a point at non zero changing voltage but zero voltage. This solution coincides with the analytical solution when the current ratio is less then one, where no voltage develops across the junction but there is a nonzero changing voltage to dissipate any voltage that may form.

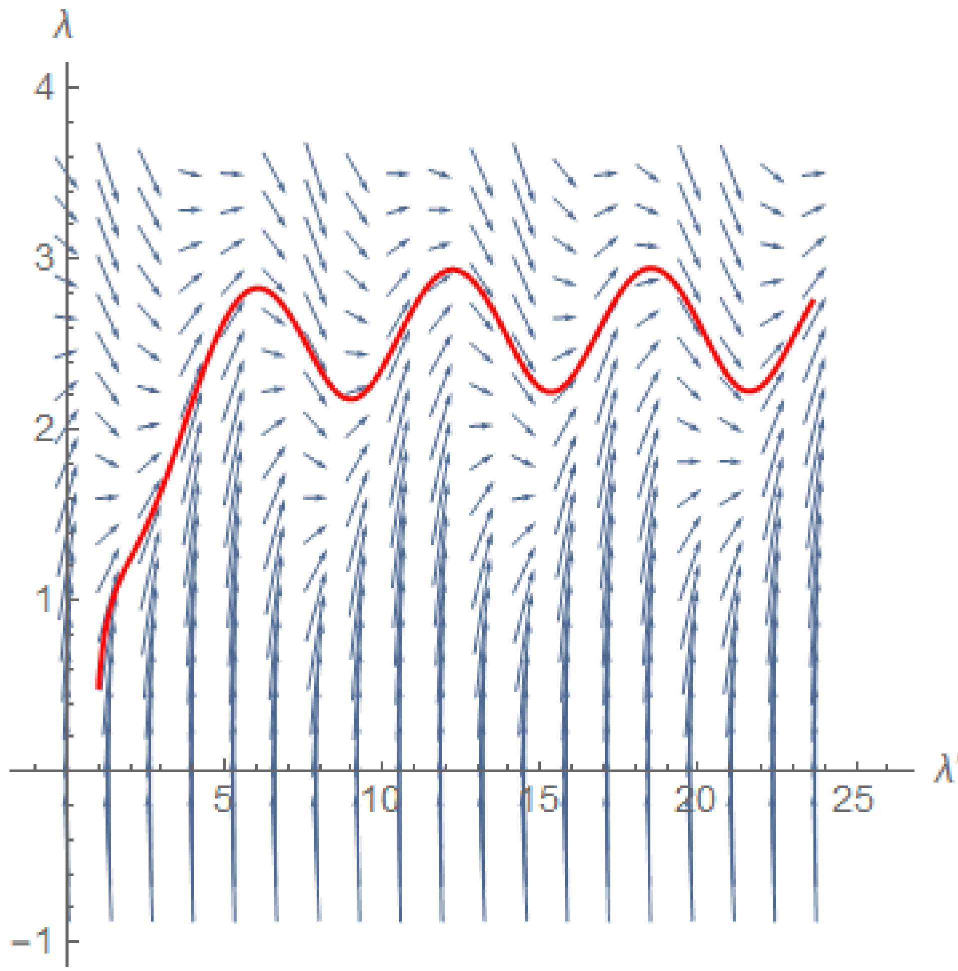

Figure 5: This solution depicts the case where the bias current is greater then the critical current. This produces a periodic solution. The voltage value that the oscillations are about depends on the value of the ratio relative to the voltage across the junction. The greater the ratio, the larger the value that the voltage oscillates around.

the directions of solutions to Equation 16. It shows the spiral sink behaviour around a zero voltage point and a nonzero change in voltage point which changes depending on the values of Q and the ratio of currents.

The solution in the second case shows the oscillating voltage which in the analytical solution was approximated using the average value of the function over one period. This oscillating voltage is a result of the constant tunneling back and forth across the junction of the Cooper pairs. If there was no applied current across the junction then the oscillations would be about the x-axis and oscillate between some positive and negative maximum voltage, but since the Josephson junction is in a circuit with an applied current, the oscillations are favored in one direction resulting in a positive overall voltage.

5 Directions for Further Research

To further the study of this model, studying the system created when eliminating the other set of constants in Equation 15 would offer a more complete picture of the Josephson junction. Eliminating the other set of constants results in a system which exhibits hysteresis as the voltage develops and is dissipated across the junction. It is also important to note that instead of focusing on the overdamped case and nondimensionalizing, Equation 15 can be solved using elliptic integrals of the first kind.

References

[1] Strogatz, Steven Henry. Nonlinear Dynamics and Chaos: with Applications to Physics, Biology, Chemistry, and Engineering. Perseus Books, 2001.

[2] Feynman, Richard. “Feynman Lectures.” The Feynman Lectures on Physics Vol. III Ch. 21: The Schrödinger Equation in a Classical Context: A Seminar on Superconductivity. 05 Apr. 2019 http://www.feynmanlectures.caltech.edu/III_21.html.

[3] You, J. Q., and Franco Nori. “Atomic physics and quantum optics using superconducting circuits.” Nature News. 29 June 2011. Nature Publishing Group. 11 May 2019 https://www.nature.com/articles/nature10122.

Appendices

Below is the Mathematica code used to produce the numerical solutions and plots shown in Figures 4 and 5.

```
h[x_, y_] = y;
j[x_, y_, V_, S_] = V*S - V*y - V*Sin[x];
ho = -1;
```

```
jo = -1.5;


Manipulate[
 soln = NDSolve[{x'[t] == h[x[t], y[t]], y'[t] == j[x[t], y[t], V, S],
     x[0] == 1, y[0] == .5}, {x[t], y[t]}, {t, 0, 10}];
 solnplot =
  ParametricPlot[{x[t], y[t]} /. soln, {t, 0, 10},
   PlotStyle -> {Thick, Red}];
 slopes =
  VectorPlot[{h[x, y], j[x, y, V, S]}, {x, 0, 25}, {y, -.5, 3.5},
   VectorPoints -> 20, Axes -> True,
   VectorScale -> {.03, Automatic, None},
   LabelStyle -> (FontSize -> 14), Frame -> False,
   AxesLabel -> {\[Lambda]', \[Lambda]}];
 Show[slopes, solnplot], {V, 1, 10}, {S, -10, 10}]
(* the voltage is on the y axis and the change in voltage is on the x \
axis*)
(*ignore all of the error messages that pop up below the manipulate \
graph, just define the functions in the first line and this should \
run like normal*)

Manipulate[
 jj = Plot[
   Piecewise[{{0, x < 1}, {A*R*Sqrt[(x)^2 - 1], x > 1}}], {x, 0, 4},
   AxesLabel -> {"\!\(\*FractionBox[\(I\), \(\(\\\ \
\)\*SubscriptBox[\(I\), \(c\)]\)]\)",
     "\!\(\*FractionBox[\(V\), \(\(\\\ \)\(R\\\ \*SubscriptBox[\(I\), \
\(c\)]\)\)]\)"}];
(*the axes lable is supposed to be (I/I_c, V/RI_c) *)
 v = Plot[x, {x, 0, 4}, PlotStyle -> Dashed ];
 Show[jj, v] , {A, 0, 12, 1}, {x, 0, 12, 1}, {R, 0, 12, 1}]
```