The Fair Mathematics of Fair Isle Knitting: A Study of Knitting Patterns

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December 13, 2017

Abstract

Fair Isle knitting is a unique knitting style that follows a set of general rules. These rules include its use of only two colors within a knit row, its simple color changes, its repeats in pattern structure, and its formation of geometric shapes through the color changes and repeated patterns used. Alongside these standards comprising what a makes a Fair Isle knitting pattern, it is important to understand that Fair Isle knitting is knit in a circle. The patterns formed are therefore cyclical. Due to the nature of Fair Isle knitting patterns, it came into question whether or not it would be possible to count the total number of pattern possibilities that can be created for various stitch lengths. In simplifying complex patterns, a series of patterns within patterns became apparent. Using the information discovered, an algorithm was formulated for finding an upper bound for the total number of potential patterns per stitch length. This study went further in developing a method for finding an approximate number of possible patterns for odd numbered stitch lengths.

1 Fair Isle Knitting

The craft of Fair Isle knitting originated on the miniscule island of Fair Isle, situated between the Atlantic Ocean and the North Sea. The specific details surrounding the origin of Fair Isle knitting are somewhat mysterious as historians debate existing evidence supporting its more romanticized beginnings verses its potentially modern development. Yet while the Fair Isle knitting narrative remains under dispute, the unique character and components of Fair Isle knitting have maintained its authenticity and established its particularity from other knitting styles. The significant traits comprising Fair Isle knitting include its use of only two colors within a knit row, its simple color changes, its repeats in pattern structure, and its formation of geometric shapes through the color changes and repeated patterns used. Another important characteristic of Fair Isle knitting pertains to its method of knitting. It is important to recognize that Fair Isle knitting is knit in a circle, indicating that the patterns created are structurally cyclical.

Considering its use of simple color changes, its repeats in pattern structure, and its cyclical composition, one can examine a single knit row and draw out various potential patterns constituting that row. This examination raises an interesting question. If each row contains stitches of only two colors and its pattern structure repeats cyclically, is it possible to count the total number of feasible patterns that can be constructed through the Fair Isle knitting style? What is more, each person who knits a Fair Isle styled product is considered to have acquired an ownership over their personally developed pattern. If restrictions are set on the stitch length of a row, can we count the number of potential patterns to be considered as a knitter's uniquely and personally designed pattern? To discover whether we can count the possible Fair Isle knitting patterns, this study will investigate knitting patterns within knitting patterns, beginning with an exploration of the patterns within singular rows of various stitch length.

2 Definitions and Development

Definition 1 A **minimal pattern** is any pattern that satisfies the following criteria:

1. Utilizes exactly two colors;
2. Begins with one of the two colors and ends with the second of the two colors, with no other color changes occurring in between;
3. Does not contain four stitches of the same color in a row.

Example 1 Consider the knitting patterns shown in Figure 1.

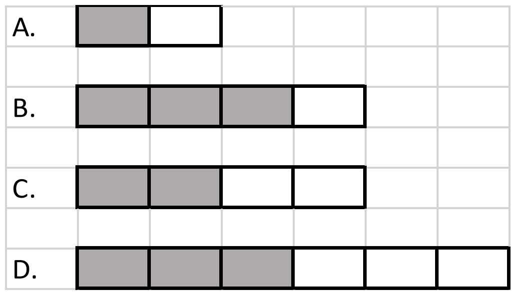

Figure 1: Minimal Patterns

Pattern A is a minimal 2 stitch pattern. Patterns B and C are both minimal 4 stitch patterns. Pattern D is a minimal 6 stitch pattern.

Example 2 In contrast, consider the knitting patterns shown in Figure 2.

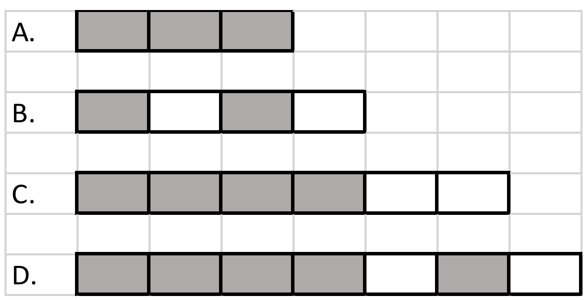

Figure 2: Non-Minimal Patterns

Pattern A is not a minimal 3 stitch pattern as there is no color change occurring. Pattern B is not a minimal 4 stitch pattern as there is more than one color change occurring. Pattern C is not a minimal 6 stitch pattern as there are four stitches in a row of the same color. Pattern D is not a minimal 7 stitch pattern as there are four stitches in a row of the same color and more than one color change is occurring.

Lemma 1 (Minimal Pattern Stitch Length) If a minimum of two colors are required to develop a minimal pattern, and there cannot be four stitches in a row of the same color, then there are five stitch lengths capable of minimal patterns. These stitch lengths are of at most six stitches and of at least two stitches.

Proof. Let n be the number of stitches within a pattern. By the definition of a minimal pattern, the pattern must contain two colors and so there must be a minimum of 2 stitches to complete a two-colored pattern. This means that the minimum number of stitches required to have a minimal pattern must be at least 2, or $n \geq 2$. Furthermore, as there cannot be 4 stitches of the same color in a row, we can have at most 3 stitches of one color in a row. Since a minimal pattern utilizes exactly two colors and there can be at most 3 stitches of the same color in a row, a minimal pattern can be of size at most 6 stitches. Therefore, the maximum number of stitches possible to create a minimal pattern is 6, or $n \leq 6$. This means that minimal patterns must be of stitch length n where $2 \leq n \leq 6$. ∎

Example 3 Consider the patterns shown in Figure 3. Patterns containing orange stitches are non-minimal and patterns containing gray stitches are minimal.

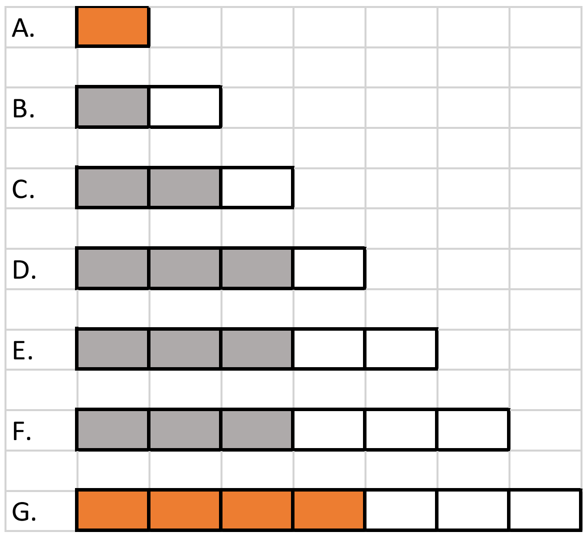

Figure 3: Minimal Pattern Stitch Length

Pattern A is not a minimal pattern as there is no color change occurring. Therefore, a single stitch cannot be a minimal pattern. Patterns B-F are all minimal patterns. Pattern G is not a minimal pattern as there would need to be at least 4 stitches of the same color in a row for a color change to occur. Therefore, a 7-stitch pattern cannot be a minimal pattern. Stitch lengths greater than 7 would similarly require at least 4 stitches of the same color in row.

Definition 2 A **sequence of minimal patterns** is any composition of minimal patterns. In this respect, the minimal patterns defined in Definition 1 can be used as building blocks to form larger patterns.

Example 4 Consider the pattern in Figure 4.

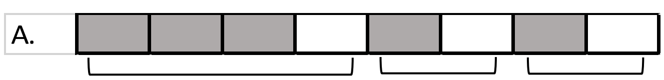

Figure 4: A Sequence of Minimal Patterns

Pattern A is not a minimal pattern, but it is a sequence of minimal patterns. Pattern A is composed of minimal patterns of stitch lengths 4, 2, and 2 respectively. Together, this sequence of minimal patterns forms an 8-stitch knitting pattern.

Definition 3 The **color inverse** of a pattern interchanges the two colors utilized in a pattern. Inverting all the colors in a pattern does not change the overall structure of the pattern. Therefore, a pattern and its color inverse will be considered the same.

Example 5 Consider the following 4 stitch pattern, Pattern A, found in Figure 5.

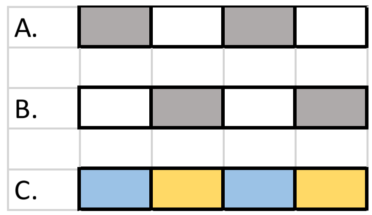

Figure 5: Color Inversion

If Pattern A is our original pattern and we invert the colors in Pattern A, we obtain Pattern B. Since any two colors can be chosen when creating a pattern, Pattern A and its inverse (Pattern B) are the same 4 stitch pattern. This interchangeability of colors is further demonstrated by Pattern C. Pattern C uses two different colors altogether, yet pattern C has the same pattern structure as Pattern A and Pattern B.

Definition 4 A **cyclic permutation** of a pattern is the shifting of the stitches of a pattern either to the right or left that maintains the structure of the pattern.

Example 6 Consider the following 4 stitch pattern, Pattern A, shown in Figure 6.

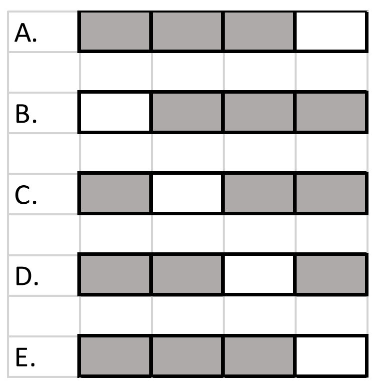

Figure 6: Cyclic Permutations of a Pattern

If Pattern A is our original pattern, Patterns B, C, and D are cyclic permutations of Pattern A. Pattern B shifts each stitch from Pattern A to the right, thereby moving the rightmost stitch to the beginning of the pattern. Therefore, Pattern B is a cyclic permutation of Pattern A in which the white stitch is moved to the front and the three black stitches are each moved one stitch to the right. Due to the cyclical nature of Fair Isle knitting, both Patterns A and B maintain the same pattern structure, but simply begin at different starting points. Patterns C and D are examples of patterns that further move each stitch rightwards. Pattern E is the result of shifting each stitch in Pattern D one more stitch to the right. Notice that Pattern E represents having shifted the stitches enough to return to the same starting point as Pattern A.

Definition 5 A **partition** is any written form of a number n as a sum of its addends. The order of the addends is not significant, but most partitions are written from largest to smallest addend.

Example 7 Consider $n = 4$. The following are partitions of n.

$$4 = (4)$$

$$4 = (3 + 1) = (3,1)$$

$$4 = (2 + 2) = (2,2)$$

$$4 = (2 + 1 + 1) = (2,1,1)$$

$$4 = (1 + 1 + 1 + 1) = (1,1,1,1)$$

Definition 6 A **partition utilizing minimal patterns** is any written form of a stitch length n as a sum of its minimal pattern stitch lengths.

Example 8 Consider the following 8 stitch pattern, Pattern A, shown in Figure 7.

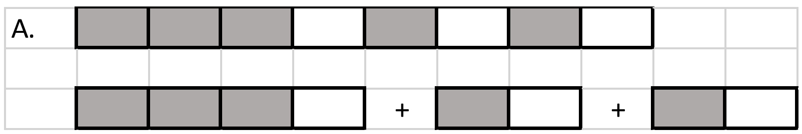

Figure 7: A Partition Utilizing Minimal Patterns

Pattern A is an 8-stitch pattern and it is broken down into its minimal pattern components comprising of one minimal 4 stitch pattern and two minimal 2 stitch patterns. In this way, the 8-stitch pattern can be written as a partition in which $n = 8 = (4 + 2 + 2) = (4,2,2)$ and each addend in the partition is a minimal pattern.

3 Results

Theorem 1 (Minimal Patterns) By the definition of a minimal pattern and Lemma 1, there exist a total of 8 minimal patterns.

Proof. As a minimal pattern must be of stitch length n where $2 \leq n \leq 6$, the number of potential minimal patterns is limited to those specific stitch lengths. By Definition 1, a minimal pattern must use two colors within the pattern, contain one color change, and cannot contain 4 stitches of the same color in a row. By this definition of a minimal pattern, we further eliminate any patterns that contain fewer or more than two colors, fewer or more than one color change, or 4 stitches of the same color in a row. By process of elimination, we find that a total of 8 minimal patterns exist. Those 8 minimal patterns are listed in Figure 8. ∎

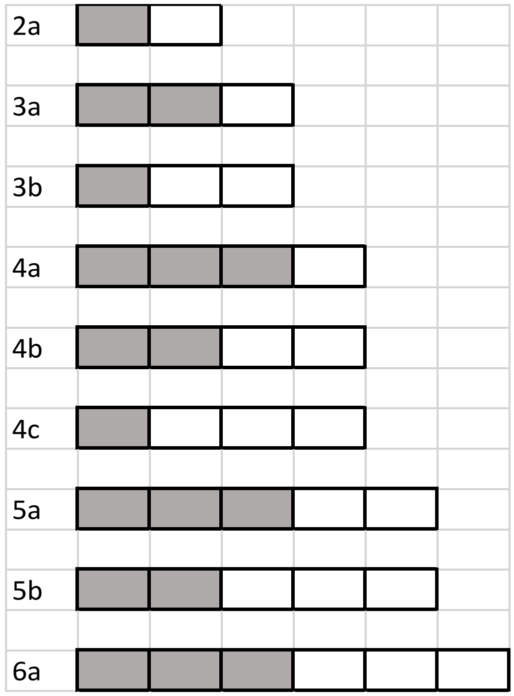

Figure 8: All Existing Minimal Patterns

Theorem 2 (Sequences of Minimal Patterns are Composed of Partitions) Every sequence of minimal patterns of stitch length n is composed of partitions utilizing minimal pattern stitch lengths.

Proof. By Definition 2, minimal patterns can be used as building blocks to compose sequences of minimal patterns. If minimal patterns must be of stitch length n where $2 \leq n \leq 6$, then all sequences are composed of minimal pattern sizes n where $2 \leq n \leq 6$. If we consider the stitch length of a sequence, then the sequence length is partitioned into addends of minimal pattern stitch lengths $2 \leq n \leq 6$. Therefore, each sequence is composed of partitions utilizing the numbers $2,3,4,5,$ and/or 6. We can then use partitions as a mechanism for breaking down sequences into their smaller and more easily countable minimal pattern components. ■

Theorem 3 (Partitions Constructing Sequences of Minimal Patterns) Partitions utilizing minimal pattern stitch lengths construct sequences of minimal patterns.

Proof. By Definition 2, minimal patterns can be used as building blocks to compose sequences of minimal patterns. Partitions utilizing minimal patterns create a pattern of greater stitch length than any of its singular partition elements. Since partitions utilizing minimal patterns construct a larger pattern composed of minimal patterns, and a sequence of minimal patterns is any composition of minimal patterns, we can use partitions of minimal patterns to build sequence of minimal patterns. Moreover, we can simplify partitions utilizing minimal patterns into numerical partitions comprising stitch lengths $2,3,4,5,$ and/or 6. We can then use these numerical partitions utilizing minimal pattern stitch lengths as a mechanism for constructing sequences of minimal patterns. ■

Theorem 4 (Arrangement of Partition Elements) The arrangement of addends, or elements, in a partition affects the number of potential sequences for that sequence length. (There can be more than one sequence created from a partition of n).

Proof. Consider a partition $A = (a, b, c)$ of some stitch length n. If $a, b,$ and c each represent a minimal pattern stitch length, then the partition lists out a sequence of minimal patterns with $a, b,$ and c combined in that order. Since the patterns of Fair Isle knitting repeat cyclically, a cyclic permutation of partition A, denoted as partition $B = (b, c, a)$, maintains the same pattern structure as partition A. Therefore, partitions A and B both represent the same pattern and can be counted as one pattern rather than as two distinct patterns. However, consider a partition that is not a cyclic permutation of partition A such as partition $C = (a, c, b)$. Since the order of the elements in the partition changes the structure of the pattern created, partitions A and C can be counted as two distinct patterns even though they are composed of the exact same elements. Therefore, the arrangement of the elements within a partition affects the number of potential patterns to be counted. ■

Example 9 Consider the partition $(4,3,2)$ of stitch length nine in Figure 9.

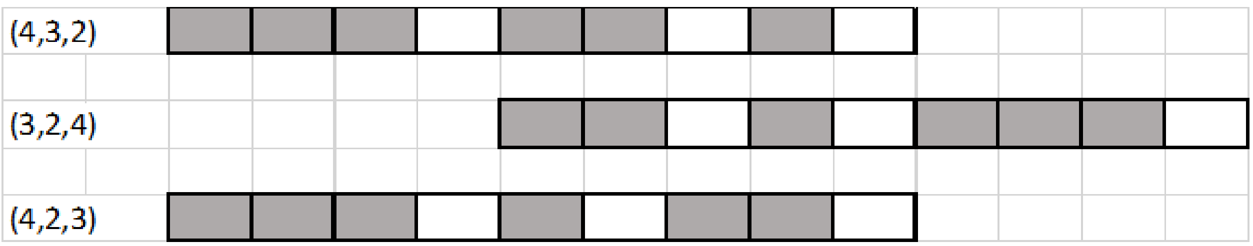

Figure 9: Partitions and Arrangement of Elements

The partitions written as $(4,3,2)$ and $(3,2,4)$ have the same pattern structure, whereas partition $(4,2,3)$ has a different pattern structure. This demonstrates how the order/arrangement of elements in a partition affects the pattern created.

Note on Partition Element Choices (Effect of Minimal Pattern Alternatives) Choices for singular elements in a partition affect counting. By Theorem 1, there are a total of 8 minimal patterns of which to use as building blocks for forming sequences. There is one 2 stitch minimal pattern and one 6 stitch minimal pattern. However, there are two 3 stitch minimal patterns, three 4 stitch minimal patterns, and two 5 stitch minimal patterns to choose from (See Figure 8). Therefore, a partition containing a $3, 4,$ and/or 5 must account for the choices for individual elements within the partition and how those options affect the overall number of sequences formed and the number of patterns counted.

Results 1 (Counting Sequences) Figure 10 shows the results for the total number of patterns that can be created for stitch lengths n where $4 \leq n \leq 9$. These results were found by drawing out all the possibilities for patterns with consideration to color inversion to avoid overcounting.

Stitch Length n	Number of Patterns
4	3
5	2
6	5

7	5
8	11
9	14

Figure 10: Results for Patterns of Stitch Lengths n where $4 \leq n \leq 9$

General Algorithm (Counting Total Possibilities of Patterns for Stitch Length n) To find an upper bound for the total number of pattern possibilities for various stitch lengths, follow the general algorithm:

1. List out all the partitions of length n that contain some or all of elements $2,3,4,5,6$ (using software such as Mathematica).
2. Examine the number of possibilities per partition element (based on the number of minimal pattern choices per stitch length).
3. Examine the number of distinct arrangements of the partition elements.
4. Multiply the number of possibilities per partition element for each element in the partition.
5. Multiply the total number of possibilities found in step 4 by the number of distinct arrangements of the partition elements.
6. Add the answers obtained in step 5 for each partition to find an upper bound for the total number of possibilities for various stitch lengths n.

Example 10 Consider stitch length $n = 5$. Using the General Algorithm, we find:

1. The partitions of 5 (found using Mathematica) that contain some or all of elements $2,3,4,5,6$ are $(5), (3,2)$.
2. The number of possibilities per partition element are listed as $(element = number\ of\ possibilities\ for\ that\ element)$.
    a. $(5) = (5 = 2)$
    b. $(3,2) = (3 = 2, 2 = 1)$
3. The number of distinct arrangements for each partition's elements are listed as $(partition) = number\ of\ distinct\ arrangements$.
    a. $(5) = 1$
    b. $(3,2) = 1$
4. Multiplying the number of possibilities per partition element for each element in the partition, we find:
    a. $(5) = 2$
    b. $(3,2) = 2 \times 1 = 2$
5. Multiplying the total number of possibilities found in step 4 by the number of distinct arrangements of the partition elements found in step 3, we find:
    a. $(5) = 2 \times 1 = 2$
    b. $(3,2) = 2 \times 1 = 2$
6. Adding the answers obtained in step 5, we find an upper bound of 4 possible patterns for stitch length $n = 5$.

Example 11 Consider stitch length $n = 11$. Using the General Algorithm, we find:

1. The partitions of 11 (found using Mathematica) that contain some or all of elements $2,3,4,5,6$ are $(6,5), (6,3,2), (5,4,2), (5,3,3), (5,2,2,2), (4,4,3), (4,3,2,2), (3,3,3,2), (3,2,2,2,2)$.
2. The number of possibilities per partition element are listed as $(element = number\ of\ possibilities\ for\ that\ element)$.
    a. $(6,5) = (6 = 1, 5 = 2)$

b. $(6,3,2) = (6 = 1, 3 = 2, 2 = 1)$
c. $(5,4,2) = (5 = 2, 4 = 3, 2 = 1)$
d. $(5,3,3) = (5 = 2, 3 = 2, 3 = 2)$
e. $(5,2,2,2) = (5 = 2, 2 = 1, 2 = 1, 2 = 1)$
f. $(4,4,3) = (4 = 3, 4 = 3, 3 = 2)$
g. $(4,3,2,2) = (4 = 3, 3 = 2, 2 = 1, 2 = 1)$
h. $(3,3,3,2) = (3 = 2, 3 = 2, 3 = 2, 3 = 1)$
i. $(3,2,2,2,2) = (3 = 2, 2 = 1, 2 = 1, 2 = 1, 2 = 1)$

3. The number of distinct arrangements for each partition's elements are listed as $(partition) = number\ of\ distinct\ arrangements$.
   a. $(6,5) = 1$
   b. $(6,3,2) = 2$
   c. $(5,4,2) = 2$
   d. $(5,3,3) = 1$
   e. $(5,2,2,2) = 1$
   f. $(4,4,3) = 1$
   g. $(4,3,2,2) = 3$
   h. $(3,3,3,2) = 1$
   i. $(3,2,2,2,2) = 1$
4. Multiplying the number of possibilities per partition element for each element in the partition, we find:
   a. $(6,5) = 1 \times 2 = 2$
   b. $(6,3,2) = 1 \times 2 \times 1 = 2$
   c. $(5,4,2) = 2 \times 3 \times 1 = 6$
   d. $(5,3,3) = 2 \times 2 \times 2 = 8$
   e. $(5,2,2,2) = 2 \times 1 \times 1 \times 1 = 2$
   f. $(4,4,3) = 3 \times 3 \times 2 = 18$
   g. $(4,3,2,2) = 3 \times 2 \times 1 \times 1 = 6$
   h. $(3,3,3,2) = 2 \times 2 \times 2 \times 1 = 8$
   i. $(3,2,2,2) = 2 \times 1 \times 1 \times 1 = 2$
5. Multiplying the total number of possibilities found in step 4 by the number of distinct arrangements of the partition elements found in step 3, we find:
   a. $(6,5) = 2 \times 1 = 2$
   b. $(6,3,2) = 2 \times 2 = 4$
   c. $(5,4,2) = 6 \times 2 = 12$
   d. $(5,3,3) = 8 \times 1 = 8$
   e. $(5,2,2,2) = 2 \times 1 = 2$
   f. $(4,4,3) = 18 \times 1 = 18$
   g. $(4,3,2,2) = 6 \times 3 = 18$
   h. $(3,3,3,2) = 8 \times 1 = 8$
   i. $(3,2,2,2,2) = 2 \times 1 = 2$
6. Adding the answers obtained in step 5, we find an upper bound of 74 possible patterns for stitch length $n = 11$.

Results 2 (Upper Bound vs. Actual Quantity Comparison)

Stitch Length n	Upper Bound for Patterns	Actual Number of Patterns
4	4	3
5	4	2
6	9	5

7	10	5
8	22	11
9	28	14
11	74	37

Figure 11: Results for Upper Bound Quantity vs. Actual Quantity of Possible Patterns

Theorem 5 (Total Hypothetical Pattern Possibilities for Odd Stitch Lengths) The number of hypothetically plausible patterns found using the general algorithm for odd stitches lengths will always be an even number.

Proof. When multiplying the number of possibilities per partition element for each element in a partition, you are multiplying any arrangement of or including the numbers 1,2, and 3. These numbers are found by the number of choices per stitch length, in which there is one 2 stitch minimal pattern and one 6 stitch minimal pattern and there are two 3 stitch minimal patterns, three 4 stitch minimal patterns, and two 5 stitch minimal patterns to choose from. To obtain an odd numbered stitch length, the elements of the partition must add to an odd number. Since adding even numbers gives an even number and adding an odd and an even number gives an odd number, there must be at least one odd numbered element in any partition of an odd numbered stitch length. Therefore, any partition of an odd numbered stitch length must contain at least one 3 or 5, as these are the only odd numbered minimal pattern stitch lengths. Furthermore, since there are two choices for 3 stitch and 5 stitch minimal patterns, in counting the number of choices per element in a partition, there would be at least one element in which there are two choices. Because the number of choices per element is multiplied by 2, it follows that the total number of possibilities is divisible by 2 and therefore even. ∎

Theorem 6 (Total Actual Pattern Possibilities for Odd Stitch Lengths) There are exactly half as many of the hypothetically countable patterns found using the general algorithm for odd stitch lengths.

Proof. Since the general algorithm gives an upper bound for the total number of pattern possibilities for various stitch lengths, we are overcounting in our consideration of those patterns. Each minimal pattern can be color inverted. In particular, minimal patterns 3a and 3b, 4a and 4c, and 5a and 5b are the color inverses of each other. When counting the number of odd stitch patterns, we count the number of patterns with a larger proportion of black stitches. For example, minimal pattern 3a contains 2 black stitches and 1 white stitch while minimal pattern 3b contains 1 black stitch and 2 white stitches. When using minimal 3 stitch patterns as building blocks for sequences of minimal patterns, we may only count the patterns containing minimal 3 stitch patterns of greater black stitch proportionality. In other words, because minimal pattern 3a contains a majority of black stitches and is the color inverse of pattern 3b, the number of patterns with majority black stitches is half of the number of hypothetical patterns found using the general algorithm. ∎

4 Conclusion and Directions for Further Research

In this study we developed an algorithm for finding an upper bound for the total number of patterns that can be created for various stitch lengths. We further discovered how to approximate the upper bound to an exact value for odd numbered stitch lengths. Using the general algorithm, we are able to find an upper bound for the total number of Fair Isle knitting patterns that can be created given a stitch length n. By simplifying complex patterns into their minimal components, analyzing the relation of the partition function in building sequences of minimal patterns, and accounting for symmetry (color inversion and cyclic permutations), we finalized a method for obtaining the total number of feasible patterns that can be constructed through the Fair Isle knitting style for odd numbered stitches lengths.

Further research is necessary to develop a method for more exactly approximating the total number of patterns for even numbered stitch lengths. However, due to the nature of the partition function and its involvement with this research, it is unlikely that an exact formula with be found.

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