Determination of Two-Dimensional Solutions to the Navier-Stokes Equations

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Abstract

The Navier-Stokes Equations describe the motion of a viscous fluid. Although they have a wide range of practical uses, finding an analytical solution is nearly impossible, except for under certain simplifications. We examine the problem of a constant unidirectional flow applied to a semi-infinite plate. Several theorems and properties, including Reynold's Transport Theorem and the Divergence Theorem, are used to derive the Navier-Stokes Equations from first principles. This is followed by a Nondimensionalization, which puts the equation in a dimensionless form, with no fixed length or time scales. We use the Similarity Solution method to introduce a change of variables, which reduces the PDE to a soluble ODE, the Blasius Equation. We solve the ODE numerically after considering various boundary conditions and assumptions. Finally, the results of the numerical solution are transformed back into the original spatial variables and discussed in terms of Boundary Layer Theory.

1 Introduction

The Navier-Stokes Equations (NSEs) are a system of partial differential equations (PDEs) that govern the motion of a viscous fluid. Named for physicists Claude Louis Navier and George Gabriel Stokes, this system of PDEs are both extremely useful in a variety engineering fields and they present an unsolved problem to mathematicians. The Clay Mathematics Institute has named the NSEs a Millennium Prize Problem, offering one million dollars for the discovery of their solution in three dimensions. Even though solving the NSEs poses such a challenge, through the use of various techniques and simplifying assumptions, they have been solved both analytically and numerically for a range of cases. This thesis will focus on a particular case that makes use of a number of these techniques. This is an application to a semi-infinite plate, which is depicted in Figure 1. In this case, the x-axis is represented as a stationary boundary and there is a constant uni-directional velocity across the y-axis. We use the term

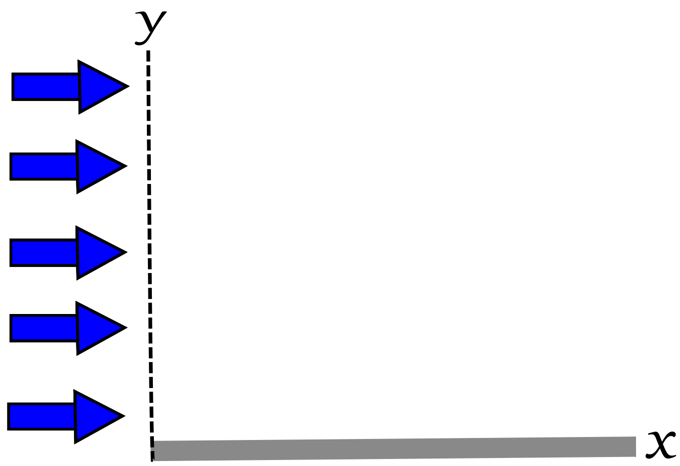

Figure 1: Semi-infinite plate with constant unidirectional inlet velocity

"semi-infinite" to illustrate that the region bounded by the positive x and y axes has no upper x and y boundaries. We also note that the scale of the x dimension, being a semi-infinite plate, is much larger than that of the y dimension. The NSEs can be used to determine the characteristics of the the flow within this region of interest. Solving the NSEs is equivalent to determining a function for the velocity in this region, which we can then analyze and visualize.

2 Definitions and Development

2.1 Derivation

The NSEs are derived from an application of Newton's Second Law of motion to a fixed region of known volume, called a *control volume.* Newton's Second law is often expressed as

$$\sum F = ma = \frac{dp}{dt},$$

which states that any changes in momentum, denoted by p, are due to some net force exerted on the object in question. Thus we say that momentum in a closed system is conserved. However, to work using a control volume and not a system of particles, which may travel beyond the boundaries of the control volume, we need a new derivative that expresses the rate of change of a quantity only within the fixed control volume. This derivative will be used to find the time dependent rate of change of mass and momentum in the control volume.

We will begin with the conservation of mass. To illustrate this approach, imagine a container of known volume, with a well-defined inlet and outlet area, such as a pipe. The mass of a fluid within this container can be expressed as a volume integral:

$$m = \iiint_V \rho \, dV,$$

where ρ is the density of the fluid and V is the volume of the bounded region. Now imagine the liquid exiting the container through a control surface with a known area, S, and vector, $\mathbf{n}$, outward-pointing normal to the surface. Then the equation for the amount of mass leaving the control volume per unit time is given as

$$\frac{dm}{dt} = \oiint_S \rho \mathbf{v} \cdot \mathbf{n} \, dS,$$

where $\mathbf{v}$ is the velocity vector. The rate of change, with respect to time, of mass within the control volume is equal to the rate at which it is exiting through the control surface, yielding

$$\oiint_S \rho \mathbf{v} \cdot \mathbf{n} \, dS = -\frac{d}{dt} \iiint_V \rho \, dV. \tag{1}$$

This equation is the integral form of Reynolds Transport Theorem. The time derivative on the right hand side can be brought inside the integral because ρ is continuous over V and, according to Leibniz's Rule, the derivative of the integral is equal to the integral of the derivative. Next we apply the Divergence Theorem to the left hand side of Equation (1), which states that for a continuous vector field $\mathbf{F}$, in a volume V, we have

$$\iiint_V (\nabla \cdot \mathbf{F}) \, dV = \oiint_S (\mathbf{F} \cdot \mathbf{n}) \, dS.$$

The Divergence Theorem can be interpreted as equating the divergence of a vector field throughout an entire volume with the flux of that vector field across the boundaries of the same volume. Thus Equation (1) is equivalent to

$$\iiint_V \rho (\nabla \cdot \mathbf{v}) \, dV = -\iiint_V \frac{\partial}{\partial t} \rho \, dV.$$

Integrating over the same volume allows us to remove the triple integrals, producing the differential form of Reynolds Transport Theorem:

$$\rho (\nabla \cdot \mathbf{v}) = -\frac{\partial \rho}{\partial t}.$$

Moving the time derivative to the left hand side gives the derivative of mass within the control volume. This is called the *material* or *LaGrangian derivative.* In the case of mass continuity, where matter cannot be created or destroyed in a closed system, the material derivative equals zero, and we have:

$$\frac{\partial \rho}{\partial t} + \rho (\nabla \cdot \mathbf{v}) = 0. \tag{2}$$

The material derivative can be applied to other physical quantities within the control volume. Now we consider the momentum continuity, where we apply the material derivative to $\rho\mathbf{v}$ instead of just ρ, converting from mass to momentum. The material derivative of momentum in a control volume is

$$\frac{\partial \rho\mathbf{v}}{\partial t} + \rho\mathbf{v}(\nabla\cdot\mathbf{v}),$$

where the x, y, and z components of the velocity are u, v, and w, respectively. The derivative of momentum, as Newton's Second Law states, is equal to the sum of the forces, now on the control volume rather than a system of particles, which is given as

$$\begin{aligned} \mathbf{F}_x &= \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right), \\ \mathbf{F}_y &= \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right), \\ \mathbf{F}_z &= \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right), \end{aligned} \tag{3}$$

where Equation (3) gives all three components of the momentum continuity equation. The relevant forces to consider are *body forces* and *surface forces.* Body forces, such as gravity, act upon the entire mass in the control volume. Surface forces, such as pressure and shear forces, act upon the specific surfaces that they interact with. For this derivation, the surface forces will be expressed in terms of stress tensors. A stress is defined as a force per unit area, which expresses the rate of deformation of a fluid. A tensor is a geometric object that contains the magnitude, direction, and plane upon which a particular force is acting [1]. See Figure 2 for a representation of the various stresses upon a control volume, denoted as σ.

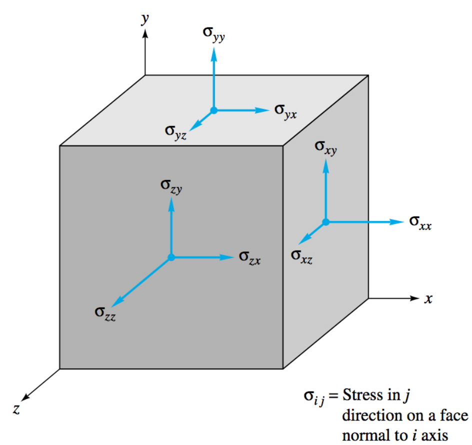

Figure 2: Stresses acting upon a control volume [2]

The i and j subscript indices represent the surface normal to the stress and the direction in which the stress is applied, respectively. The tensor form of the stresses is given as

$$\sigma_{ij} = \begin{bmatrix} -p + \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & -p + \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & -p + \tau_{zz} \end{bmatrix},$$

where i and j denote the rows and columns. Pressure is given as p and τ_{ij} represents the viscous stress terms, together they comprise the surface forces in a viscous fluid. Notice that the pressure terms only act in the same direction as the surface that has a normal vector in that direction. We will proceed in the x-direction only, because the y and z surface forces are computed using the same process. The net force due to these stresses is not given by the values for the stresses themselves, but rather their differences or gradients, giving rise to the differential expression of the change in surface forces:

$$dF_{x,\text{surf}} = \left(\frac{\partial}{\partial x}\sigma_{xx} + \frac{\partial}{\partial y}\sigma_{yx} + \frac{\partial}{\partial z}\sigma_{zx}\right) dxdydz.$$

We now substitute in the values for σ_{ij} given in the tensor above:

$$\frac{dF_x}{dV}_{\text{surf}} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}. \tag{4}$$

This substitution for the x, y, and z components reveals that the surface forces can be written in vector notation as

$$\frac{dF}{dV}_{\text{surf}} = -\nabla p + (\nabla \cdot \tau_{ij}),$$

where τ_{ij} represents only the viscous stress terms in the tensor, σ_{ij}. Combining Equations (3) and (4) yields our equation for the x-component of the momentum conservation equation in stress tensor form:

$$\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right), \tag{5}$$

where ρg_x is the x-component of gravity, our only body force. The next step is to write the surface force terms as a function of the other variables that we have introduced. For this, we introduce the definition of a *Newtonian* fluid, in which "the viscous stresses, at every point, are linearly proportional to the rate of change of [the fluid's] deformation over time.[3]" This assumption about the nature of the fluid allows us to substitute in derivatives of the velocity components for the viscous stress terms. The substitutions that are to be made are given below [2]:

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} \qquad \tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$
$$\tau_{yy} = 2\mu\frac{\partial v}{\partial y} \qquad \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$
$$\tau_{zz} = 2\mu\frac{\partial w}{\partial z} \qquad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

Applying these variable substitutions to Equation (5) yields the standard form of the incompressible Navier-Stokes Equations for a Newtonian fluid. They are second order and nonlinear, which will be discussed later. Below is the vector form for the whole system as well as the full form of the x, y, and z-components:

$$\rho\mathbf{g} - \nabla p + \mu\nabla^2\mathbf{V} = \rho\frac{d\mathbf{V}}{dt}, \tag{6}$$

$$\begin{aligned}
\rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) &= \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right),\\
\rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) &= \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right),\\
\rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) &= \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right).
\end{aligned} \tag{7}$$

The right hand side of Equation (6) is a compact way of denoting the material derivative, which can be expanded in full partial derivative notation as in Equation (7). For convention, because volume will not be referenced in further sections, velocity will be denoted as $\mathbf{V}$, with components u, v, and w.

2.2 Nondimensionalization

To analyze the semi-infinite plate application, we will consider the two dimensional case of Equation (6). We will apply a technique called *nondimensionalization*, which introduces dimensionless *reference variables* in order to extract a dimensionless form of the equations. These reference variables are fixed quantities of the same units as our original variables. Our dimensionless variables, designated by the star notation, are defined as ratios of the original variables with their corresponding reference variable, given as: $p^* = \frac{p}{\rho U^2}$, $u^* = \frac{u}{U}$, $x^* = \frac{x}{L}$, $y^* = \frac{y}{\delta}$, $t^* = \frac{tU}{L}$, $g^* = \frac{gL}{U^2}$, $v^* = \frac{vL}{U\delta}$. Here U is the reference inlet velocity, L and δ are the x and y reference lengths, respectively, and t is the reference time interval. All other variables are rewritten as combinations of these variables and various parameters. It is worth noting that the application to the semi-infinite plate implies that the x-reference quantity, L, will be relatively large compared to the y-reference length, δ.

We begin with the two-dimensional form of Equation (7) and will substitute our dimensionless variables into the equation in order to extract a dimensionless form. Notice that right hand side of the equation below contains the total time derivative, where Equation (7) gives the expanded form containing the component partial derivatives. We will use this process for the x-component:

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \rho\frac{du}{dt},$$

$$\rho\frac{g^* U^2}{L} - \frac{\partial(p^* \rho U^2)}{\partial(x^* L)} + \mu\left(\frac{\partial^2(u^* U)}{\partial(x^* L)^2} + \frac{\partial^2(u^* U)}{\partial(y^* \delta)^2}\right) = \rho\frac{d(u^* U)}{d(t^* L/U)}.$$

We then factor the reference quantities and parameters out of each term:

$$\frac{\rho U^2}{L}g^* - \frac{\rho U^2}{L}\frac{\partial p^*}{\partial x^*} + \frac{\mu U}{L^2}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{L^2}{\delta^2}\frac{\partial^2 u^*}{\partial y^{*2}}\right) = \frac{\rho U^2}{L}\frac{du^*}{dt^*}. \tag{8}$$

Multiplying both sides of Equation (8) by $\frac{L^2}{\mu U}$ yields

$$\frac{\rho U L}{\mu}g^* - \frac{\rho U L}{\mu}\frac{\partial p^*}{\partial x^*} + \left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{L^2}{\delta^2}\frac{\partial^2 u^*}{\partial y^{*2}}\right) = \frac{\rho U L}{\mu}\frac{du^*}{dt^*}. \tag{9}$$

All but one term contains a factor of $\rho UL/\mu$. This is the expression for the dimensionless quantity called the *Reynolds Number*. The Reynolds Number, often denoted as Re, is important for determining whether a flow can be classified as laminar, corresponding to a low Reynolds Number, or turbulent, corresponding to a high Reynolds Number. As many turbulent flow scenarios do not have analytical solutions, we will concentrate on laminar flows. With this in mind, we will make a few simplifying assumptions based on order of magnitude or "scale", neglecting terms that have much lower relative contributions to the behavior of the fluid than others. We reason that our fluid exhibits negligible spatial variation in pressure and negligible body forces compared to the force of the inlet velocity of the system. This allows us to cancel the $\partial p^*/\partial x^*$ and g^* terms. These assumptions are reasonable because a two dimensional laminar flow, which has no depth dimension, experiences minor gravitational force and is likely to exhibit constant pressure throughout, and so the force of the inlet velocity will dominate the flow of the system, rendering these other terms insignificant. The terms with the second order derivatives of velocity also have different relative scales. As stated before, in the case of the semi-infinite plate, the length L is much greater than δ, meaning that the ratio L^2/δ^2 is much greater than 1. In the majority of real world flows, the Reynolds Number is often much larger than 1, even for small-scale laminar flows. For example, the average Reynolds Number for blood flow in the human body is between 100 and 1000 [4]. Therefore, we will assume that L^2/δ^2 is on the same order of magnitude as Re, and thus the x derivative is negligible compared to the y derivative. These points will reduce Equation (9) considerably to

$$Re \frac{\partial^2 u^*}{\partial y^{*2}} = Re \frac{du^*}{dt^*}.$$

Finally, we divide by Re and expand the time derivative in terms of the components of u, which is a function of x, y, and t. Using the Chain Rule, we know that $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$, where $\frac{dx}{dt} = u$ and $\frac{dy}{dt} = v$, the velocity components themselves. The fully nondimensionalized and scaled form of Equation (7) is thus

$$\frac{\partial^2 u^*}{\partial y^{*2}} = u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*}. \tag{10}$$

The equation has been reduced to a dimensionless form with no time dependence and only spatial dependence. Going forward, we will not use the star notation, it will be understood that we are working strictly with dimensionless forms of the variables. Notice that Equation (10) expresses the relationship between two independent variables, u and v, which are each functions of two spatial dependent variables, x and y. Of course, we cannot solve one differential equation for two distinct functions. This motivates the next analytical technique that we will apply.

2.3 Similarity Solution

The *Similarity Solution* technique introduces a change of variables that will transform Equation (10) into a differential equation involving one function of one independent variable, while not sacrificing the characteristic nature of the physical system that is being modeled. This will produce an equation that can be solved with analytical or numerical techniques. We introduce a *stream function*, that expresses both spatial derivatives in terms of one variable. Let ψ denote the stream function where

$$\frac{\partial \psi}{\partial y} = u \qquad \text{and} \qquad -\frac{\partial \psi}{\partial x} = v.$$

This change of variables can be substituted into Equation (10), which yields

$$\frac{\partial^3 \psi}{\partial y^3} = \frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial y \partial x} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2}. \tag{11}$$

We now have Equation (11), which is a third order differential equation of one dependent variable. We give an expression for ψ in order to introduce another change of variables and transform Equation (11) into an equation of only one independent variable:

$$\psi = x^a f(\eta) \qquad \text{and} \qquad \eta = x^b y.$$

The function, f, is called the *similarity function* and is in terms of a single variable, η. The rationale behind expressing ψ and η as we have involves choosing quantities that contain x and y, which are the relevant variables of our system, and expressing them in a form that allows us to reduce our governing PDE to a simpler ordinary differential equation (ODE) in terms of one variable that encompasses x and y. To illustrate this concept, we will substitute ψ and η into Equation (11). This substitution, when simplified reduces to

$$x^{2a+2b-1}\left((a+b)f'\left(x^by\right)^2 - af\left(x^by\right)f''\left(x^by\right)\right) = x^{a+3b}f'''\left(x^by\right). \tag{12}$$

Although the dependence on y alone has been eliminated from Equation (12), the expression still has a dependence on x alone. We will determine suitable values for a and b that are consistent with the physics in the semi-infinite plate application. First, in order to remove the x dependence, we set $2a+2b-1 = a+3b$, which will give both sides of Equation (12) the same exponent on x. The second condition on a and b comes from a particular boundary condition of the system. The application to the semi-infinite plate assumes a horizontal velocity of zero at the surface of the plate, or the $y = 0$ region. This is called the *no slip boundary condition*, and it can be used to add a constraint on a and b. Understanding that at $y = 0$, the x-component of the velocity, u, is zero, allows us to apply both sets of variable changes to the boundary conditions:

$$u_{y=0} = 0 \implies \left.\frac{\partial\psi}{\partial y}\right|_{y=0} = 0 \implies \frac{\partial}{\partial y}x^a f(x^by) = 0 \implies x^{a+b}f'(x^by) = 0.$$

The no slip condition requires that we set $a+b = 0$ so that we have $f'(0) = 0$. Under this condition, we have two equations for our two unknowns and can solve for a and b. Solving this system of linear equations yields $a = 1/2$ and $b = -1/2$ and updates Equation (12) to

$$x^{-1}\left(-\frac{1}{2}f\left(\frac{y}{\sqrt{x}}\right)f''\left(\frac{y}{\sqrt{x}}\right)\right) = x^{-1}f'''\left(\frac{y}{\sqrt{x}}\right). \tag{13}$$

We can divide the left and right hand sides of Equation (13) by x^{-1}, producing the ordinary differential equation,

$$-\frac{1}{2}f(\eta)f''(\eta) = f'''(\eta). \tag{14}$$

The variable η expresses the relationship between the spatial variables as $y/\sqrt{x}$. It is worth noting that Equation (14), also called the *Blasius Equation*, is a third order ODE, whereas Equation (11) is only second order, and both remain nonlinear. Just as with algebraic equations, introducing nonlinearity into the problem of finding solutions makes solving differential equations much more difficult. Because of this, we are not able to find an analytical solution to Equation (14), although we can approximate the solution effectively using numerical techniques.

2.4 Numerical Solution

In order to solve Equation (14), a third order ODE, we require three conditions on the equation. The first condition, which we have discussed, is the no slip condition. We have defined that at the surface of the plate, the fluid exhibits no horizontal velocity. This boundary condition, when applied to the stream function, ψ, is transformed to give $f'(0) = 0$, a condition in terms of f. The remaining boundary conditions must be converted from constraints in terms of u and v to constraints in terms of $f(\eta)$.

The second boundary condition also comes from the no slip condition. This requires that the vertical velocity at the surface of the plate, or $y = 0$, also be set to 0. To differentiate this condition from the no slip condition for horizontal velocity, we will call this the *bottom boundary condition*. We can apply this condition to the stream function as before, computing

$$v_{y=0} = 0 \implies -\frac{\partial \psi}{\partial x}\bigg|_{y=0} = 0 \implies -\frac{\partial}{\partial x}\bigg|_{y=0} x^a f(x^b y) = 0 \implies$$
$$ax^{a-1}\Big(x^b y f'(x^b y) - f(x^b y)\Big)\bigg|_{y=0} = 0 \implies -ax^{a-1} f(x^b y) = 0.$$

Thus we have transformed the second boundary condition on the system in terms of the similarity solution variables to constrain Equation (14). This boundary condition requires that we have $f(0) = 0$ to be valid for all values of x. Finally, the last boundary condition on the semi-infinite plate system, called the *top boundary condition*, expresses the opposite effect of the no slip condition. Specifically, at the top boundary of our system, the horizontal velocity of the fluid will be equal to the inlet velocity of the fluid. This means that at some distance from the surface of the plate, the stationary effects of the plate will be fully overcome. Although this system does not have a well-defined top boundary, it can be said that as y approaches infinity, the horizontal velocity of our fluid approaches the inlet velocity, which has been defined as our reference velocity. Since we are working in terms of dimensionless variables, which are expressed as ratios of our dimensional variables divided by some reference quantity, the dimensional horizontal velocity, u, approaches the reference velocity, U, and so the dimensionless horizontal velocity, u/U, approaches 1. From our transformation of u in the previous section, we can then state that as y approaches infinity, η approaches infinity, and we know that $f'(\eta)$ approaches 1. To summarize, the three conditions on Equation (14) are:

- No Slip: at $y = 0$, $f'(0) = 0$,
- Bottom: at $y = 0$, $f(0) = 0$, and
- Top: as $y \to \infty$, $f'(\eta) \to 1$.

With these conditions representing the boundary conditions upon our system, we can apply a numerical solution technique to Equation (14), which we can rewrite as

$$f'''(\eta) + \frac{1}{2} f(\eta) f''(\eta) = 0.$$

This form of Equation (14) can be solved using Mathematica for a single function, f, of a single variable, η. Although η does not contain any relevant information about the system by itself, when we express η as $y/\sqrt{x}$, we can use these results to draw conclusions about the behavior of the fluid in terms of the x and y coordinates for the semi-infinite plate application.

3 Results

The result of the numerical solution technique from Mathematica is a discrete solution, consisting of a series of points that define the function, $f(\eta)$. These points are given in the form of an *interpolating function*, which allows us to approximate the value of $f(\eta)$ for any value of η. As a sample of the solution, below is a table containing the approximate values of $f(\eta)$ and its derivatives for a number of different η values:

η	$f(\eta)$	$f'(\eta)$	$f''(\eta)$
0	$3.3087e-24$	$-1.3553e-20$	0.332057
0.5	0.0414928	0.165885	0.330907
1	0.165572	0.32978	0.323008
1.5	0.370139	0.486789	0.30258
2	0.650024	0.629766	0.266752
2.5	0.996311	0.75126	0.217411
3	1.39681	0.846044	0.16136
3.5	1.8377	0.91304	0.107773
4	2.30575	0.955518	0.0642342
4.5	2.79013	0.979514	0.0339799
5	3.28327	0.991542	0.0159068
5.5	3.78057	0.996879	0.00657866
6	4.27962	0.998973	0.00240205
6.5	4.77932	0.999699	0.000774173
7	5.27924	0.999922	0.000220236
7.5	5.77922	0.999982	0.0000552616
8	6.27921	0.999996	0.0000122401

To begin, we must examine the dependence of f upon η. Figure 3 shows this relationship and the long term behavior of $f(\eta)$. While $f(\eta)$ does not have any clear physical significance, the quantity that does have clear physical significance is $f'(\eta)$, which represents the horizontal velocity, u. This reasoning can be shown from the variable transformation that was used in Section 2.3 to explain the no slip boundary condition, with the understanding that $a = 1/2$ and $b = -1/2$:

$$u = \frac{\partial \psi}{\partial y} \implies \frac{\partial}{\partial y} x^a f\left(x^b y\right) \implies x^{a+b} f'\left(\frac{y}{\sqrt{x}}\right) \implies u = f'(\eta).$$

We have shown that the u-velocity can be expressed in terms of f, with the independent variable, η. Although η does not have its own physical meaning,

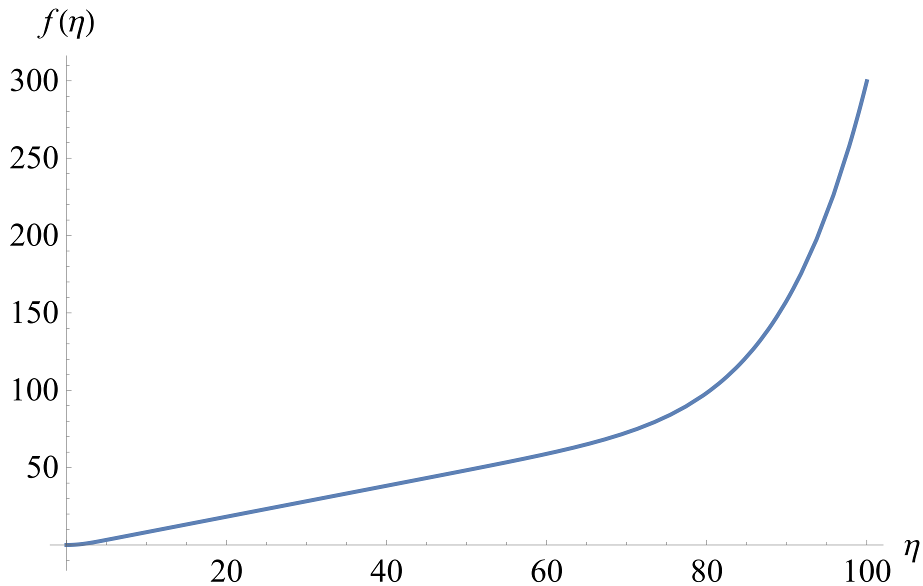

Figure 3: The similarity function f vs. η

it does represent $y/\sqrt{x}$, which is how we will analyze it. Figure 4 shows the behavior of $f'(\eta)$. As we may expect, at some value of η, the horizontal velocity approximately reaches the inlet velocity, and so $f'(\eta)$ will approach 1. This is the point where the dimensionless horizontal velocity equals the reference velocity. As Figure 4 shows, the dimensionless horizontal velocity asymptotes towards 1. The interpolating values in the table confirms this behavior and shows that while f' will never reach 1, because the top boundary condition is given as a limit as η approaches infinity, it comes within 99 percent of 1 at approximately $\eta = 5$. We say at $\eta = 5$, the u-velocity has reached 99 percent of the *free stream velocity*, effectively overcoming the no slip condition at the surface of the plate. This result has been confirmed by experimentation on numerous occasions. The real significance of this results is that at some distance above the plate, corresponding to $\eta = 5$, the flow reaches 99 percent of the free stream velocity, and this region is called the *boundary layer*. The boundary layer has a shape given by $\eta = y/\sqrt{x}$ where $\eta = 5$, or $y = 5\sqrt{x}$. Figure 5 is a contour plot in x and y coordinates where the black line is the shape of the boundary layer and the contour shows the u-velocity profile within this region. Again, we can see that the horizontal velocity at the plate is zero, and it increases as we move in the positive y-direction. The next step is to find the formula for determining the y-component of the velocity, v. We will use the variable transformation method that was used for the bottom boundary condition, now

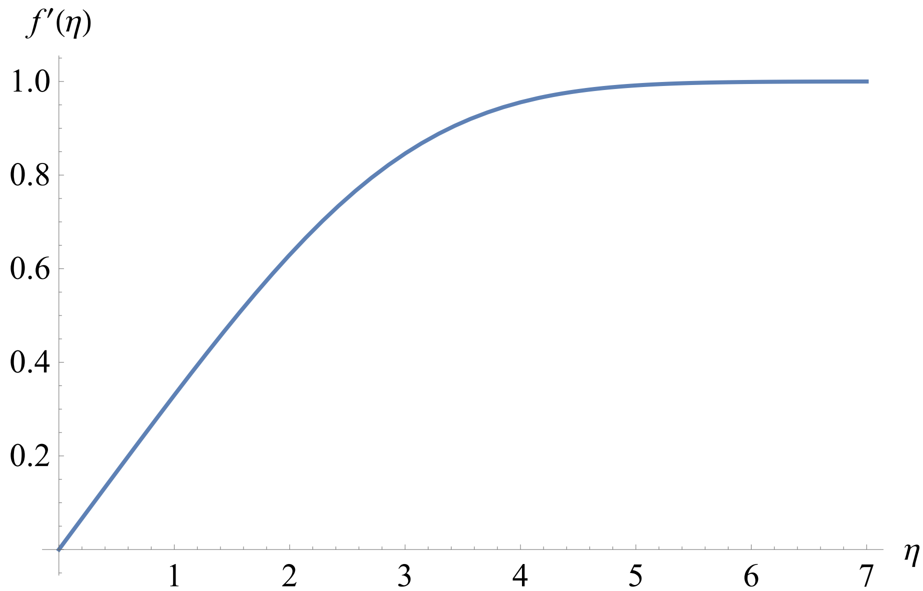

Figure 4: The similarity function f' vs. η

using the computed values for a and b:

$$v = -\frac{\partial \psi}{\partial x} \implies -\frac{\partial}{\partial x} x^a f\left(x^b y\right) \implies \frac{1}{2\sqrt{x}}\left(\frac{y f'\left(\frac{y}{\sqrt{x}}\right)}{\sqrt{x}} - f\left(\frac{y}{\sqrt{x}}\right)\right)$$
$$\implies \frac{1}{2\sqrt{x}}\Big(\eta f'(\eta) - f(\eta)\Big).$$

The function for the v-velocity is much more complicated than the u-velocity, containing a combination of f, η, and x alone. Even so, we can use Mathematica to plot a contour of the vertical velocity profile along with the boundary layer shape. Figure 6 shows that the vertical velocity is nearly 0 throughout most of the region below the boundary layer. Moving in the positive y-direction shows a gradual increase in the v-velocity. However, the results of this plot are less clear at the lower x-values because the factor of $1/2\sqrt{x}$ causes the solution to quickly approach infinity. Despite this fact, we can clearly interpret that the flow shows only a small amount of vertical velocity throughout the region of interest, but it increases with the increase in y. This result is confirmation of Bernoulli's Principle, which states that where there is a gradient in the velocity of a fluid, there is a resulting pressure gradient, which causes an increased flow in the direction orthogonal to the original flow [5]. Bernoulli's Principle is the driving force behind flight in airplanes. The relatively high velocity over the top of an airfoil, as opposed to the lower velocity below, causes a pressure gradient that produces air velocity upward, which we call *airlift*. These u and v velocity

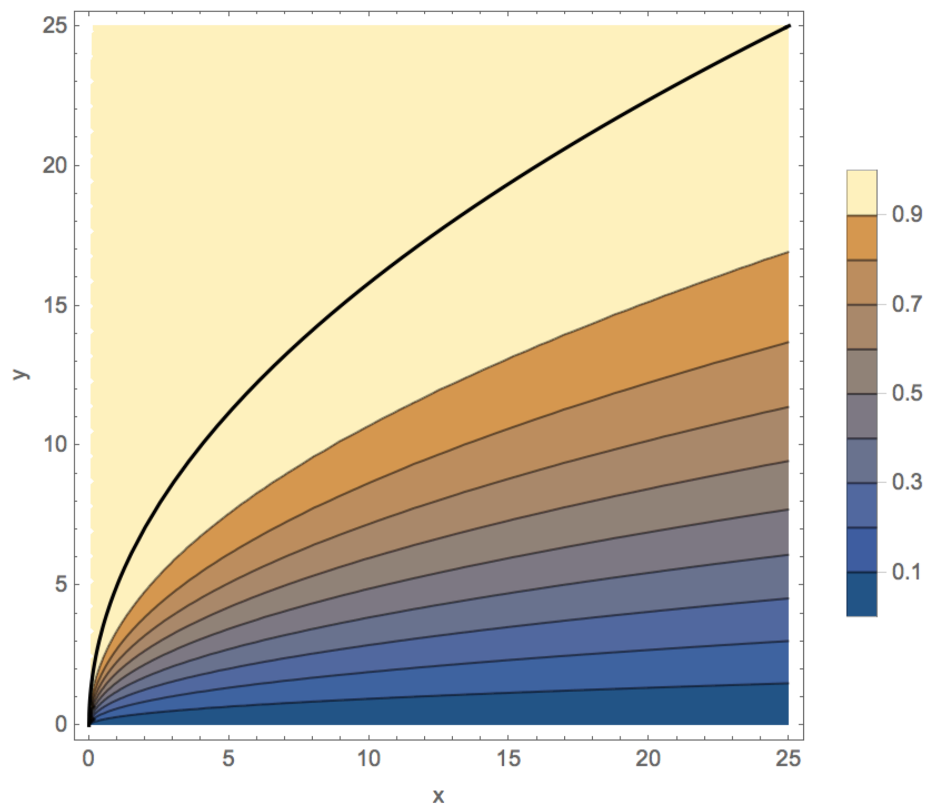

Figure 5: Contour plot of the u-velocity and the boundary layer

profiles are effectively the solutions to the two-dimensional NSEs that have been recovered from a single function, $f(\eta)$.

4 Conclusion and Directions for Further Research

As we have shown, through using analytical methods such as nondimensionalization and similarity solutions, the NSEs can be transformed into expressions that have a reduced number of unknown variables. These analytical methods, specifically nondimensionalization, have a very broad range of applications to problems that have a large dependence on several variables or have quantities of many units involved. However, these analytical techniques cannot ultimately provide the solution to this problem due to the nonlinearity of the NSEs. In this situation, numerical methods can be used to effectively approximate the solutions to this system, with the appropriate boundary conditions. Historically, many of the major advances that have been made in the field of fluid dynamics have come as a result of the computational approach to solving the

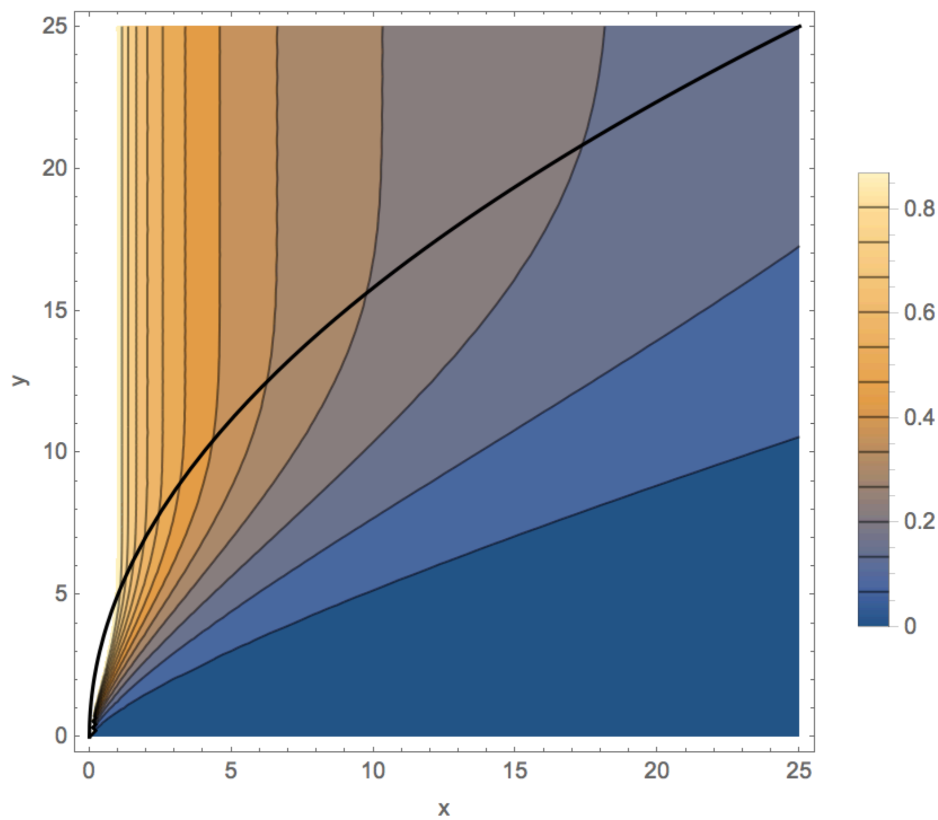

Figure 6: Contour plot of the v-velocity and the boundary layer

NSEs. The field of computational fluid dynamics is an incredibly rich topic, with many approaches to solving the NSEs in their full PDE form, such as Finite Difference and Finite Volume Method, just to name a few. Large scale fluid simulations have become an essential tool in many industrial and academic settings, including weather simulation, computer-generated video graphics, and commercial development of fluid-like products. Today, the greatest challenges that face researchers in fluid dynamics are building more efficient models for fluid simulation and understanding the inherently chaotic nature of the NSEs. While classical results, like the one derived in this thesis, are very well understood, there are still a huge amount of unanswered questions and potential areas for future research.

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