Dynamical Systems and Circle Maps

Steven Metallo
Carthage College
smetallo@carthage.edu

May 5, 2014

Abstract

Dynamics is a branch of mathematics that studies how systems change with time, and this can be done using function iteration or differential equations. Our focus is on the dynamics of the circle map function, $f_n(x) = nx \mod 1$, where $n \in \mathbb{N}$ and the domain is the interval $[0, 1]$ where $0 \equiv 1$. This simple function leads to complicated dynamics; it has periodic points of every period as well as infinitely many aperiodic points. We introduce symbolic dynamics by using a Markov partition to split up the domain into intervals with specific properties. For Markov partitions, we construct a Markov matrix and prove, through matrix conjugation, that the eigenvalues of the corresponding Markov matrices are 2 and roots of unity.

1 Introduction

A *dynamical system* is any system that changes with time. As a few examples, consider the stock market, the daily temperature, and solutions to differential equations. The first two examples are *discrete* dynamical systems while the latter is a *continuous* dynamical system. The system we will study has less complicated dynamics than these examples, but this enables us to gain insight into more complicated systems. By working with an easy system we develop methods and tools to use when exploring complicated dynamical systems.

2 Definitions and Development

Before we begin our investigation of the dynamical system, we list some definitions and introduce the circle map function. Let $f_n : [0,1] \to [0,1]$ with

$$f_n(x) = nx \mod 1,$$

where n is a natural number. One method of investigating the dynamics of this system is iterating the circle map function:

$$f_n^0(x) := x,$$

$$f_n^1(x) = nx \mod 1,$$
$$f_n^2(x) = n^2 x \mod 1,$$
$$\vdots$$
$$f_n^i(x) = n^i x \mod 1.$$

Let us now introduce some definitions to aid the development of the analysis of our analysis.

Definition 1. A point is **periodic** with period k if there exists an $k \in \mathbb{N}$ such that

$$f_n^k(x) = x.$$

Conversely, a point is **aperiodic** if there is no such k.

The notion of periodic and aperiodic points enables us to understand the complexity of our system, and this is seen in the following claim.

Claim 2. The doubling map, $f_2(x)$ has periodic points of every period as well as infinitely many aperiodic points.

Proof. We shall first prove that there are periodic points of every period. Let us choose $x_0 = 1/(2^t - 1) \in [0,1]$. Then, applying function iteration we get

$$f_2^i(x_0) = \frac{2^i}{2^t - 1} \mod 1.$$

Now if $i < t$, then we have $f_2^i(x_0) < 1$, so suppose $i = t$. We notice

$$f_2^t(x_0) = \frac{2^t}{2^t - 1} \equiv \frac{1}{2^t - 1} \pmod 1 = x_0,$$

so x_0 has period t.

Next we prove that there are infinitely many non-period points for the doubling map. Suppose there are no aperiodic points. Let a be an irrational number in the domain. Then, by assumption, we have

$$f_2^k(a) = 2^k a \mod 1 = a,$$

for some $k \in \mathbb{N}$. This implies, however, that

$$a(2^k - 1) = m,$$

where $m \in \mathbb{N}$, by the definition of modular arithmetic. However, since a is irrational, then $a(2^k - 1)$ is also irrational. This is a contradiction, so we conclude that a is aperiodic. Furthermore, since there are an infinite number of irrational points in $[0, 1]$, we conclude that there are an infinite number of aperiodic points. □

We learn from the claim that the simple function $f_2(x) = 2x \mod 1$ has vastly complicated dynamics. It has both infinitely many periodic and non-periodic points.

Even though our system is highly complicated, we can still investigate it. One such method is through the orbit of a point.

Definition 3. The **orbit** of a point p is the set of points $\{f_n^i(p)|i = 0, 1, 2, ...\}$.

As an example of this definition, we use the doubling map to compute the orbit of $\frac{1}{5}$.

Example 4. We compute directly

$$f_2^0\left(\frac{1}{5}\right) = \frac{1}{5},$$
$$f_2^1\left(\frac{1}{5}\right) = \frac{2}{5},$$
$$f_2^2\left(\frac{1}{5}\right) = \frac{4}{5},$$
$$f_2^3\left(\frac{1}{5}\right) = \frac{8}{5} \mod 1 \equiv \frac{3}{5},$$
$$f_2^4\left(\frac{1}{5}\right) = \frac{16}{5} \mod 1 \equiv \frac{1}{5}.$$

Thus, the orbit of $\frac{1}{5}$ is

$$\left\{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\right\}.$$

We also note that the orbit length is 4.

We could use a more graphical representation of the orbit of $\frac{1}{5}$. If we use an arrow to signify the next iterate of the function, then we can represent the orbit as

$$\frac{1}{5} \to \frac{2}{5} \to \frac{4}{5} \to \frac{3}{5} \to \frac{1}{5}.$$

Since we have discussed the first few definitions we are ready to introduce a slightly more complicated method of investigating the dynamics of our doubling function. Through *symbolic dynamics* we can investigate the dynamics of the circle map using matrices and partitions instead of using function iteration. We use a special type of partition called a Markov partition.

Definition 5. A **Markov partition** of $[0, 1]$ for $f_2(x)$ is a partition $P_m = \{I_1, I_2, ..., I_m\}$ such that that

1. $\mathring{I}_i \cap \mathring{I}_j = \emptyset$ if $i \neq j$ or $I_i \cap I_j$ is at most one point, and
2. if $f_2(I_i)$ intersects I_j at more than one point, then $f_2(I_i)$ covers I_j completely.

The I_i are called the intervals of the Markov partition.

The Markov partition does not need to have sub-intervals of equal length, but for our system we only consider such Markov partitions. In exploring our system we use the Markov partition with m sub-intervals of equal length

$$P_m = \left\{ \left[0, \frac{1}{m}\right], \left[\frac{1}{m}, \frac{2}{m}\right], \left[\frac{2}{m}, \frac{3}{m}\right], ..., \left[\frac{m-1}{m}, 1 \equiv 0\right] \right\}.$$

We use the Markov partition to construct the Markov matrix, and this is done by applying the $f_2(x) = 2x \mod 1$ to the intervals of the Markov partition to discern the covering relations. After we have mapped the elements of the Markov partition, we construct the Markov matrix to introduce symbolic dynamics. Thus, instead of using function iteration to investigate the dynamics of our system, we introduce a Markov partition and a corresponding Markov matrix. Let us explore the idea of Markov partitions by working through the example of P_5.

Example 6. Let P_5 be the Markov partition of the unit interval with five sub-intervals. Then we have

$$P_5 = \left\{ \left[0, \frac{1}{5}\right], \left[\frac{1}{5}, \frac{2}{5}\right], \left[\frac{2}{5}, \frac{3}{5}\right], \left[\frac{3}{5}, \frac{4}{5}\right], \left[\frac{4}{5}, 1 \equiv 0\right] \right\}.$$

Applying the function to the intervals yields

$$\begin{aligned}
f_2(I_1) &= \left[0, \frac{2}{5}\right] = I_1 \cup I_2, \\
f_2(I_2) &= \left[\frac{2}{5}, \frac{4}{5}\right] = I_3 \cup I_4, \\
f_2(I_3) &= \left[\frac{4}{5}, \frac{1}{5}\right] = I_5 \cup I_1, \\
f_2(I_4) &= \left[\frac{1}{5}, \frac{3}{5}\right] = I_2 \cup I_3, \\
f_2(I_5) &= \left[\frac{3}{5}, 0\right] = I_4 \cup I_5.
\end{aligned}$$

This gives us the covering relationships

$$\begin{aligned}
I_1 &\to \{I_1, I_2\}, \\
I_2 &\to \{I_3, I_4\}, \\
I_3 &\to \{I_1, I_5\}, \\
I_4 &\to \{I_2, I_3\}, \\
I_5 &\to \{I_4, I_5\}.
\end{aligned}$$

Finally, we take the covering relations and construct the Markov matrix corresponding to P_5. We signify the covering by a 1 and a non-covering by a 0. Thus we get the matrix

$$M_5 = \begin{array}{c} \\ I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{array} \begin{array}{c} \begin{array}{ccccc} I_1 & I_2 & I_3 & I_4 & I_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}. \tag{1}$$

We see that (1) has some astonishingly nice properties. The row sum is 2 because after we apply f_2 to the intervals of P_5, their lengths double. Furthermore, because the Markov partition covered the entire domain of our function, the column sum is 2.

Next we demonstrate constructing the Markov partition with four sub-intervals, using P_4.

Example 7. We proceed in the same manner as in Example 6 to identify the covering relationships

$$\begin{aligned} I_1 &\to \{I_1, I_2\}, \\ I_2 &\to \{I_3, I_4\}, \\ I_3 &\to \{I_1, I_2\}, \\ I_4 &\to \{I_3, I_4\}. \end{aligned}$$

Using these covering relationships we get

$$M_4 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \qquad (2)$$

These Markov matrices come in many flavors, but we only concern ourselves with two. First, as in (1), if the Markov partition has an odd number of sub-intervals, then the matrix will have a middle row which will always cover the first and last sub-interval. Additionally, if the Markov partition has an odd number sub-intervals, then there are no duplicate rows in the matrix. Conversely, as in (2), if the number of sub-intervals of the Markov partition is even, then each row will be duplicated exactly twice. This matrix has the same properties as M_5 because the row sum and column sum are still 2. However, matrices constructed from Markov partitions with an even number of sub-intervals have less complicated dynamics than those constructed from an odd number of sub-intervals because the of the row duplication.

3 Results

Now that we have formulated how to construct a Markov matrix, we state our main theorem.

Theorem 8. *Let the Markov partition P_m, where $1/m$ is periodic with period $m-1$, have an odd number of intervals with equal length with boundary points $\{0, \frac{1}{m}, \frac{2}{m}, ..., \frac{m-1}{m}\}$. If this Markov partition is used to generate a Markov matrix, then the Markov matrix will have 2 and the $(m-1)^{st}$ roots of unity as its eigenvalues.*

The following proof will use the fact that similar matrices have the same eigenvalues in addition to the fact that permutation matrices have roots of unity as eigenvalues.

Proof. We begin by identifying the general form of the Markov matrix generated by our Markov partition, P_m. Note m is odd. Since we are working with the doubling map, we know that each interval covers two others. The Markov partition which we are using enables us to easily state the three types of covering relationships for the intervals

$$I_1 = \left[0, \frac{1}{m}\right] \to \left[0, \frac{2}{m}\right] = \{I_1, I_2\},$$
$$I_2 = \left[\frac{1}{m}, \frac{2}{m}\right] \to \left[\frac{2}{m}, \frac{4}{m}\right] = \{I_3, I_4\},$$
$$\vdots$$
$$I_n \to \{I_{2n-1}, I_{2n}\}, 1 \leq n \leq \frac{m-1}{2}.$$

The $\frac{m+1}{2}$ interval covers $\{I_1, I_m\}$, and the covering relations for the rest of the intervals is

$$I_{(m+3)/2} \to \{I_2, I_3\},$$
$$I_{(m+5)/2} \to \{I_4, I_5\},$$
$$\vdots$$
$$I_n \to \{I_{2n \pmod m}, I_{2n+1 \pmod m}\}, \frac{m+3}{2} \leq n \leq m.$$

Now that we have the covering relations, we can proceed to generate the

Markov matrix

$$M_m = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ & & \vdots & & \vdots & & \\ 1 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 1 & 1 & 0 & \dots & 0 & 0 \\ & & \vdots & & \vdots & & \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \end{pmatrix}. \tag{3}$$

Instead of determining the form of the characteristic equation for M_m, we construct a similar matrix using matrix conjugation, $C^{-1}M_mC$ where C is the conjugation matrix. Before we introduce C, however, we first make the following claim.

Claim 9. We claim that M_m is similar to the matrix

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ & & \vdots & & & \vdots & & \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ & & \vdots & & & \vdots & & \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

Proof. We begin by listing the general forms of the conjugation matrix and the inverse matrix,

$$C = \begin{pmatrix} 1 & -1 & 0 & 0 & \dots & 0 \\ 1 & 1 & -1 & 0 & \dots & 0 \\ 1 & 0 & 1 & -1 & \dots & 0 \\ 1 & 0 & 0 & 1 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ & & \vdots & \vdots & & \\ 1 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}, \tag{4}$$

and

$$C^{-1} = \frac{1}{m}\begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ -m-1 & 1 & 1 & 1 & \dots & 1 \\ -m-2 & -m-2 & 2 & 2 & \dots & 2 \\ -m-3 & -m-3 & -m-3 & 2 & \dots & 3 \\ -m-4 & -m-4 & -m-4 & -m-4 & \dots & 4 \\ & & \vdots & \vdots & & \\ -1 & -1 & -1 & -1 & \dots & m-1 \end{pmatrix}. \tag{5}$$

We look at the multiplication of $M_m C$ first. Since the first column of C is all 1s and the row sum of M_m is 2, the first column of $M_m C$ must be all 2s. Next, when multiplying the i^{th} row of M_m to the $j > 1$ column of C we get -1 when $j = 2i + 1 \mod m$ and a 1 when $j = 2i - 1 \mod m$. This yields the matrix

$$M * C = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ 2 & 0 & 1 & 0 & -1 & \dots & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ & & \vdots & & \vdots & & & \\ 2 & 0 & 0 & 0 & 0 & \dots & 0 & -1 \\ 2 & -1 & 0 & 0 & 0 & \dots & 0 & 1 \\ 2 & 1 & 0 & -1 & 0 & \dots & 0 & 0 \\ & & \vdots & & \vdots & & & \\ 2 & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

Next we multiply $M_m C$ by C^{-1} on the left. First we note that the row sum of C^{-1} is 0 except for the first row which has row sum 1. Thus, the first column of $C^{-1} M_m C$ consists of all 0s except for the first entry which is 2. Additionally, since the first row of C^{-1} is all 1s, the -1s in $M_m C$ become 0s. Finally, when multiplying the i^{th} row of C^{-1} by the $j > 2$ column of $M_m C$

we get 1s when $j = 2i - 1 \mod m$.

$$C^{-1}MC = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ & & \vdots & & & \vdots & & \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ & & \vdots & & & \vdots & & \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

□

We have proved our claim, and all we need to do to prove our theorem is to show that $C^{-1}M_mC$ has 2 and the roots of unity. We can treat $C^{-1}M_mC$ as a block matrix and easily see that the eigenvalue of the upper block is 2. The lower block matrix is a permutation matrix, and we claimed that permutation matrices have roots of unity as eigenvalues. Furthermore, since we assumed $1/m$ is periodic with period m, the permutation matrix has the $(m-1)^{st}$ roots of unity. □

4 Conclusion and Directions for Further Research

We started with analyzing the doubling map using function iteration, and we learned that this simple system has an infinite number of periodic points and aperiodic points. We ended with a proof that the eigenvalues of Markov matrices, constructed from Markov partitions where $\frac{1}{m}$ has period $m-1$, are 2 and $(m-1)^{st}$ roots of unity. After investigating the sequence of such $m-$values with the OEIS website, we determine the sequence number is $A077510$.

This will enable us to identify further m values for which our theorem holds. We have thoroughly demonstrated that a simple system can lead to drastically complicated dynamics, but we reiterate why we choose to study such systems: to learn about more complicated dynamics of more delicate systems.

With this project, there have been a lot of interesting questions we would like to investigate. An idea we did not pursue in this project is the idea of Markov graphs. Instead of using the covering relations of the Markov partitions to generate a matrix, we could generate a graph. As an example, consider the covering relations corresponding to the covering map and the Markov partition P_5. The Markov graph could be arranged so that it is planar, that is the graph lies in a plane. We briefly investigated this problem; however, we did not determine if the Markov graph generated from P_{11} is planar.

Another area we overlooked is how we constructed our Markov partition. We could construct a Markov partition generated by one orbit. We used P_5 as an example because $\frac{1}{5}$ is periodic with period 4. Consider the orbit of $\frac{1}{7}$,

$$\frac{1}{7} \to \frac{2}{7} \to \frac{4}{7} \to \frac{1}{7}.$$

We then use the corresponding Markov partition,

$$P = \left\{ \left[\frac{1}{7}, \frac{2}{7}\right], \left[\frac{2}{7}, \frac{4}{7}\right], \left[\frac{4}{7}, \frac{1}{7}\right] \right\}.$$

This enables us to produce the covering relations,

$$\begin{aligned} I_1 &\to \{I_2\}, \\ I_2 &\to \{I_3\}, \\ I_3 &\to \{I_1, I_2, I_3\}. \end{aligned}$$

These covering relations produce the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}.$$

This matrix is just one of the many flavors we did not investigate throughout this project. There are many more Markov partitions that could be investigated, and these investigations will enable us to better understand the dynamics of this system.

References

[1] M. Stafford, *Markov Partitions for the Doubling Map*. Ph.D. thesis, Northwestern University, 1989.

[2] M. Snavely, 2014, Complex Variables, From author's Manuscript, 2014.

[3] G. Strang, *Introduction to Linear Algebra*. Wellesley-Cambridge Press, 2003.

[4] N.J.A. Sloane, On-Line Encyclopedia of Integer Sequences, 28, April 2014, available at https://oeis.org/A077510.