Symbolic Dynamics over Free Groups

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Abstract

Symbolic dynamics is used to model functions acting on complex topological spaces. This work explores the symbolic dynamics over a non-Abelian group, the free group on two generators. We begin by introducing the fundamentals of symbolic dynamics and Cayley graph coloring. We then explain the relationship between these fields and characterize classes of periodic points of the full shift over the free group.

1 Introduction

The field of dynamical systems examines any system which changes over time. Two goals of dynamics are determining ways in which this system may exhibit predictable behavior and characterizing its chaotic behavior. Applications of dynamical systems include modeling weather patterns, population growth, molecular behavior, market behavior, and many other changing systems.

Problems arise with even simple dynamical systems. For example, if we use differential equations to model a continuous system, solving the system of equations becomes increasingly difficult as our model becomes more accurtate. The field of symbolic dynamics attempts to combat this problem. It allows us to analyze these complicated systems in general by discretizing both time and space [4]. Symbolic dynamics has various applications in linear algebra, data transmission and storage, and ergodic theory.

This work focuses on connecting periodic points of the full shift, Cayley graphs, and the free group. We examine basic properties of the shift space over one and two dimensional Cayley graphs, and focus on characterizing shifts of the free group on two generators. By focusing on a generalization of symbolic dynamics, this work aims to provide a new perspective on these basic properties of symbolic dynamics.

2 Definitions and Development

In this section, we define the shift space along with a few of its basic properties and provide examples which illustrate these definitions. Assuming a background in group theory, we briefly discuss group properties in order to thoroughly define Cayley graphs. Finally, the free group and its Cayley graph are discussed in detail.

Definition 1. The **shift space** is a finite set of symbols, $S = \{0, 1, ..., (s-1)\}$, together with an operation, σ. We define the **shift transformation** $\sigma : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ as $\sigma(x_i) = x_{i-1}$ [2].

The elements or words in the shift space are bi-infinite sequences made from the elements of the finite set S. Examples 2 and 3 illustrate the shift function for two different shift spaces.

Example 2. In $\{0,1\}^{\mathbb{Z}}$ if $x = ...10.1010...$, then $\sigma(x) = ...01.0101...$, and $\sigma(\sigma(x)) = ...10.1010....$

Example 3. In $\{0,1,2\}^{\mathbb{Z}}$ if $y = ...012.012012...$, then $\sigma(y) = ...120.120120....$

From this point, we exclusively examine bi-infinite sequences from a shift space on two symbols, $\{0, 1\}$.

Definition 4. For some $x \in \{0,1\}^{\mathbb{Z}}$, x is a **fixed point** if $\sigma(x) = x$ [3].

Definition 5. For some $x \in \{0,1\}^{\mathbb{Z}}, k \in \mathbb{N}$, x is a **periodic point of period** k if $\sigma^k(x) = x$. Conventionally, we say that x is **periodic of prime period** k if k is the smallest natural number such that $\sigma^k(x) = x$ [3].

Example 6. Let $x = ...01.0101....$ Since $\sigma(\sigma(x)) = x$, x is a periodic point of prime period 2.

Example 7. If $y = ...001.001001...$, since $\sigma^3(y) = y$, y has prime period 3.

Similarly, we see that $p_1 = ...00.000...$ and $p_2 = ...11.111...$ are the only fixed points in $\{0,1\}^{\mathbb{Z}}$.

Definition 8. Let G be a nonempty set together with a binary operation. Then $(G, \cdot)$ is a **group** if and only if the following properties hold.
1. Closure: For all elements $a, b \in G$, $a \cdot b \in G$.
2. Associativity: For all elements $a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
3. Identity: There exists an element $e \in G$ such that $\forall\ a \in G$, $a \cdot e = e \cdot a = a$.
4. Inverses: For all elements $a \in G$, $\exists\ b \in G$ such that $a \cdot b = b \cdot a = e$ [1].

Definition 9. An element $a \in G$ in the group G, is a **generator** of G if $G = \{a^n \mid n \in \mathbb{Z}\}$. The group generated by a is denoted as $G = \langle a \rangle$ [1].

Some groups have a **generating set**. In this case, more than one generator is needed to generate the entire group. This is denoted as $\langle a_1, a_2, \ldots, a_n \rangle\ |a_i \in G$. For example, $\langle (1,0), (0,1) \rangle$ generates the group $\mathbb{Z} \times \mathbb{Z}$.

Definition 10. Let G be a group and let S be a set of generators for G. We define the **Cayley graph** of G with generating set S as follows.
1. Each element of G is a vertex of the graph.
2. For $x, y \in G$, there is an edge from x to y if and only if $xs = y$ for some $s \in S$ [1].

Example 11. Figure 1 represents the Cayley graph of $\mathbb{Z}_3$. The directed graph illustrates the result of the product $1 \cdot g, \ \forall \ g \in \mathbb{Z}_3$, for the generator $1 \in \mathbb{Z}_3$.

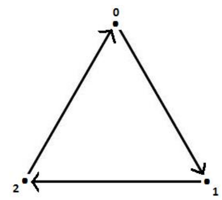

Figure 1: $\mathbb{Z}_3$ Cayley graph

Example 12. Similarly, Figure 2 represents the Cayley graph of $\mathbb{Z}_5$. The graph illustrates the result of the product $1 \cdot g, \ \forall \ g \in \mathbb{Z}_5$, for the generator $1 \in \mathbb{Z}_5$.

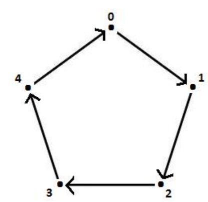

Figure 2: $\mathbb{Z}_5$ Cayley graph

Example 13. Figure 3 is a finite representation of the Cayley graph of $\mathbb{Z} \times \mathbb{Z}$ with respect to the generators $\langle(1, 0)\rangle$ and $\langle(0, 1)\rangle$. Note that its Cayley graph is the familiar integer latice.

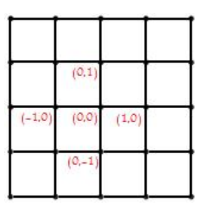

Figure 3: $\mathbb{Z} \times \mathbb{Z}$ Cayley graph

Definition 14. For a Cayley graph G and a finite set $D = \{0, 1, ..., (n-1)\}$, a **graph coloring** of G is one way of labeling the vertices of G with elements in D.

Example 15. A colored Cayley graph of $\mathbb{Z}$ is represented in Figure 4. Note that adding -1 to every integer is another way to represent the shift transformation of $\{0, 1\}^{\mathbb{Z}}$.

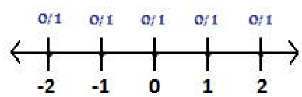

Figure 4: Colored $\mathbb{Z}$ Cayley graph

Definition 16. The **free group on two generators**, denoted as $\mathbb{F}_2 = \langle a, b \rangle$, consists of all finite words in $\{a, b, a^{-1}, b^{-1}\}$ under concatination with no relations other than inverses.

Remark 17. We define $A := a^{-1}$, $B := b^{-1}$, and $w \in \mathbb{F}_2$.

Example 18. Note that the free group is a non-abelian group and every word in the free group is reduced. Thus $abaBA \in \mathbb{F}_2$; however, because $BaAB = B^2$ in reduced form, $BaAB \notin \mathbb{F}_2$, but $B^2 \in \mathbb{F}_2$. Also $abBA = e$, where e is the identity of $\mathbb{F}_2$.

Example 19. The Cayley graph of $\mathbb{F}_2$ is represented in Figure 5[1]. The elements a, b, ab, ba have been labeled at their respective vertices to illustrate how to interpret the graph.

3 Results

The main result of this work characterizes shifts on $\{0, 1\}^{\mathbb{F}_2}$. Before stating these theorems, we first give examples of colored graphs of $\mathbb{F}_2$ and explain the shift space over the free group on two generators. Thus, for each word, $w \in \mathbb{F}_2$, the symbol at w - either a 0 or 1 - is moved to either Aw or Bw.

To color our $\mathbb{F}_2$ Cayley graph, we assign elements from our finite set $\{0, 1\}$ to each vertex in the Cayley graph, each of which represents an element of $\mathbb{F}_2$. We define the shift functions over $\mathbb{F}_2$ by multiplying every word in $\mathbb{F}_2$ by the inverses A or B of either generator. See Figure 6 and Figure 7 for an example of a colored free group.

Definition 20. The two element shift space over the free group on two generators, denoted $\{0, 1\}^{\mathbb{F}_2}$, is the colored Cayley graph of $\mathbb{F}_2$, such that the shift trasformation is defined as either $\sigma_a(w) = Aw$ or $\sigma_b(w) = Bw$.

[1]This image is derived from http://en.wikipedia.org/wiki/File:F2_Cayley_Graph.png.

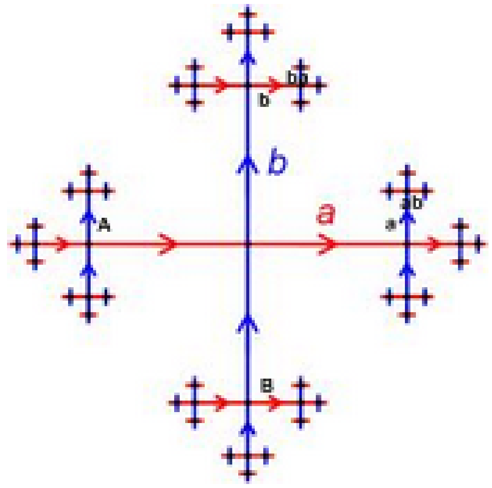

Figure 5: $\mathbb{F}_2$ Cayley graph

Definition 21. A free group coloring has **weak periodicity** if it is periodic with respect to exactly one generator. That is, either $\sigma_a(w) = Aw$ or $\sigma_b(w) = Bw$.

Definition 22. A free group coloring has **strong periodicity** if it is weakly periodic with respect to both generators. That is $\sigma_a(w) = Aw$ and $\sigma_b(w) = Bw$.

Example 23. The free group coloring in Figure 6 has strong periodicity of prime period 2. By definition, it is weakly periodic of periodicity 2 with respect to both A and B. Note that shifting by either A or B yields the coloring in Figure 7. For Figure 6, we see that shifting by A shifts every coloring to the left by one element. Repeat this same shift and the free group returns to its original coloring. Thus, it is weakly periodic of prime period 2 with respect to A. Similarly, shifting by B yields the graph in Figure 7, and shifting by B again brings the graph back to the coloring in Figure 6. Hence, this coloring is weakly periodic of prime period 2 with respect to B. Thus, by definition of strong periodicity, the coloring in Figures 6 and 7 have strong periodicity.

Theorem 24 (Weak Prime Period k A-shift)**.** *Let $w \in \mathbb{F}_2, n \in \mathbb{Z}$ and $k \in \mathbb{N}$. Then a free group coloring of the following form has weak prime k-periodicity with respect to the generator a.*

For all $w \in \{a^{kn}bw\} \cup \{a^{kn}Bw\} \cup \{a^{kn}\}$ color the vertex 0;

for all $w \notin \{a^{kn}bw\} \cup \{a^{kn}Bw\} \cup \{a^{kn}\}$ color the vertex 1.

Proof. First, assume that $\mathbb{F}_2$ has this coloring. We will show that its Cayley graph has weak prime k-periodicity with respect to the generator a. Let $P = \{a^{kn}bw\} \cup \{a^{kn}Bw\} \cup \{a^{kn}\}$. Then since our shift σ_a is representated by A left multiplication, we have $\sigma_a^k(a^{kn}bw) = A^k a^{kn}bw = a^{k(n-1)}bw \in \{a^{kn}bw\} \subseteq P$.

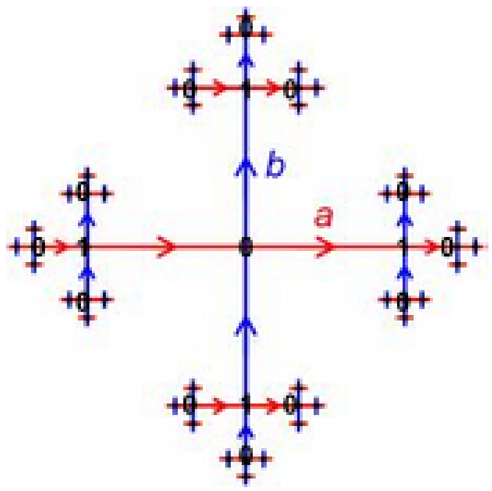

Figure 6: $\mathbb{F}_2$ Strong Periodicity

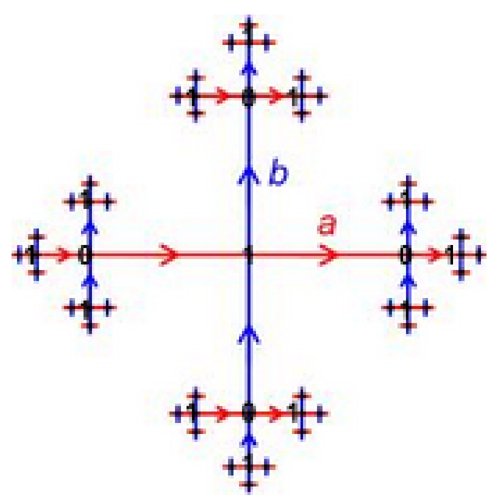

Figure 7: Shifted $\mathbb{F}_2$ Strong Periodicity

Similarly, $\sigma_a^k(a^{kn}Bw) = A^k a^{kn}Bw = a^{k(n-1)}Bw \in \{a^{kn}Bw\} \subseteq P$. Finally, we find $\sigma_a^k(a^{kn}) = A^k a^{kn} = a^{k(n-1)} \in \{a^{kn}\} \subseteq P$. Now to show that for $k > 1$ the coloring does not produce a fixed point, let $w \notin P$. Then because w is contained in the compliment of P, $\sigma_a^k(w) \notin \{a^{kn}bw\}$. Similarly, $\sigma_a^k(w) \notin \{a^{kn}Bw\}$, and $\sigma_a^k(w) \notin \{a^{kn}\}$. Therefore, this coloring has weak k-periodicity with respect to the generator a. □

Example 25. For a weak prime periodicity 3 shift with respect to A, the words $\{a^3,\, a^6b,\, a^9BabA^2, \dots\}$, and others similar to these would be colored with a 0.

Theorem 26 (Weak Period k B-shift). *Let $w \in \mathbb{F}_2, n \in \mathbb{Z}$ and $k \in \mathbb{N}$. Then a free group coloring of the following form has weak prime k-periodicity with respect to the b generator.*

For all $w \in \{b^{kn}aw\} \cup \{b^{kn}Aw\} \cup \{b^{kn}\}$ color the vertex 0;
for all $w \notin \{b^{kn}aw\} \cup \{b^{kn}Aw\} \cup \{b^{kn}\}$ color the vertex 1.

Proof. The proof of this result is similiar to the proof of the weak prime period k A-shift Theorem. □

Example 27. Similarly, for a weak prime periodicity 3 shift with respect to B, the words $\{b^3,\, b^6a,\, b^9Ab^2A^2, \dots\}$, and others similar to these would be colored with a 0.

Theorem 28 (Strong Prime Period k shift). *Let $w \in \mathbb{F}_2, n, m \in \mathbb{Z}$ and $i, j, k \in \mathbb{N}$. Then a free group coloring of the following form has strong prime k-periodicity.*

For all $w \in \{a^{kn_1}b^{km_1}a^{kn_2}b^{km_2}...a^{kn_i}b^{km_j}\} \cup \{b^{km_1}a^{kn_1}b^{km_2}a^{kn_2}...b^{km_j}a^{kn_i}\}$ color the vertex 0;

for all $w \notin \{a^{kn_1}b^{km_1}a^{kn_2}b^{km_2}...a^{kn_i}b^{km_j}\} \cup \{b^{km_1}a^{kn_1}b^{km_2}a^{kn_2}...b^{km_j}a^{kn_i}\}$ color the vertex 1.

Proof. First, assume that $\mathbb{F}_2$ has the defined coloring.
Let $R = \{a^{kn_1}b^{km_1}a^{kn_2}b^{km_2}...a^{kn_i}b^{km_j}\} \cup \{b^{km_1}a^{kn_1}b^{km_2}a^{kn_2}...b^{km_j}a^{kn_i}\}$. Then by definition of weak prime periodicity, since $R \subseteq \{a^{kn}bw\} \cup \{a^{kn}Bw\} \cup \{a^{kn}\}$, and $R \subseteq \{b^{kn}aw\} \cup \{b^{kn}Aw\} \cup \{b^{kn}\}$, this coloring is both weakly prime k A-periodic and weakly prime k B-periodic. Thus, by definition of strong periodicity, the coloring of $\mathbb{F}_2$ has strong k periodicity.

Now, let the graph of $\mathbb{F}_2$ have strong prime k periodicity. Then e is colored either 0 or 1. Without loss of generality, assume that e is colored 0. Then because it is weakly periodic with respect to the generator a, we know that a^k is colored and A^k is also colored 0. Also, a^{2k} and A^{2k} are colored 0, so by induction, $\{a^{kn}\}$ is colored 0. Similarly, since the coloring is weakly periodic with respect to the generator b, we know by induction that $\{b^{kn}\}$ is colored 0 as well. We see that if w is colored 0 off of either axis, that the entire branch becomes weakly periodic with respect to the other generator. Thus by induction, the set R is colored 0. Now, assuming the coloring is not a fixed point, assign the remaining uncolored $w \in \mathbb{F}_2$ coloring 1. Therefore this coloring is unique upto trivially switching the sets colored 0 and 1 respectively. $\square$

Remark 29. Recall Example 21. We see that Theorem 3 now applies. The vertices colored 0 are of the form $\{a^{2n_1}b^{2m_1}a^{2n_2}b^{2m_2}...a^{2n_i}b^{2m_j}\}\cup$ $\{b^{2m_1}a^{2n_1}b^{2m_2}a^{2n_2}...b^{2m_j}a^{2n_i}\}$, and the vertices colored 1 are of the form $\{a^{2n_1}b^{2m_1}a^{2n_2}b^{2m_2}...a^{2n_i}b^{2m_j}\} \cup \{b^{2m_1}a^{2n_1}b^{2m_2}a^{2n_2}...b^{2m_j}a^{2n_i}\}$. Thus this coloring is a strong prime period 2 shift.

Example 30. Theorem 26 states that we can define a strong prime period 3 shift. To do so, words of the form $\{a^3b^3,\ a^3,\ b^3,\ b^6a^3b^9a^3, \ldots\}$, among others would be colored 0. Thus, by the Strong Period k Theorem, if we continue coloring in this manner, we will produce a strong prime period 3 shift.

Example 31. Similarly, we can define a strong period 2 shift which has a fixed B shift, and a period 2 A shift. To do so, we color the words $\{e,\ a^2,\ a^{-2},\ b,\ b^{-1},\ a^2b,\ a^4b^3\ 0, \ldots\}$. We complete this coloring by concatenating powers of $a^{2m}, m \in \mathbb{Z}$, and $b^n, n \in \mathbb{Z}$. This yields a strong period 2 shift.

4 Conclusion and Directions for Further Research

We have shown that shifts over the free group can be characterized by strong and weak periodicity. The Weak Prime Period k Shift Theorems define two ways of finding and defining a weak shift over $\mathbb{F}_2$. Similarly, the Strong Prime Period k Shift Theorem defines a unique characterization of strong period k shifts. By characterizing periodic points over Cayley graphs, we have provided a more general perspective on traditional symbolic dynamics.

Because the Weak Prime Period Theorems do not define a unique characterization, one area of further research would be to find all such characterizations. Another direction for further work is to examine similar shift properties with a larger set of colors. Finally, we could examine other topological properties, such as mixing, topological mixing, and entropy to draw closer connections to the field of dynamics in general.

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