Generalization of Riemann's Rearrangement Theorem

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Abstract

In this paper, we will examine Riemann's rearrangement theorem for sequences of real numbers. Continuing on, we will briefly discuss previous generalizations of the theorem to the n dimensional case. Because of the difficulty of that proof, we present a more approachable proof for the case of complex numbers. Our proof is not significantly more difficult than the proof for real numbers. Furthermore, key aspects of the proof can be applied to the n dimensional case. From there, we are able to prove the n dimensional case using elementary techniques. We conclude by considering the possibility of an infinite dimensional generalization through sequence or function spaces.

1 Introduction

Riemann's rearrangement argument is a well known result of real analysis. It states that any conditionally convergent series of real numbers can be rearranged to converge to any real number, or to diverge. This is fairly easy to show, and the proof is very understandable. However, when attempting to generalize the proof to the more general space $\mathbb{R}^n$, problems arise. Due to the interconnectedness of the components, the same method Riemann used cannot be repeated. In this paper, we generalize Riemann's argument to $\mathbb{C}$, and equivalently $\mathbb{R}^2$. We find that rather than only having two possibilities for rearrangements, as in the real case, there are three different possibilities for the set of points the rearranged series can have as a limit.

When dealing with real numbers, there are two possibilities for the set of points that a convergent series can be rearranged to have as a limit. If the series is absolutely convergent, all rearrangements will converge to the same value. If the series is conditionally convergent, for any real number there is a rearrangement of the series that converges to that value. When working with complex numbers, however, another possibility occurs. While all rearrangements of an absolutely convergent series of complex numbers will still converge to the same value, not all complex numbers can be achieved by rearranging any series of complex numbers. It is possible for all rearrangements of a series of complex numbers to lie on a single line, rather than to be the entire complex plane. In this paper, we show that these are the only possibilities, and establish how to determine which possibility a sequence falls into.

2 Definitions and Development

To begin, we must first define a series.

Definition 1. A series of complex numbers $(\sum z_n)$ is said to converge to some number a if given any $\epsilon > 0$ there exists some $M \in \mathbb{N}$ such that for any $m \geq M$, $|\sum_{n=1}^{m}(z_n) - a| < \epsilon$. If $(\sum |z_n|)$ also converges the series is *absolutely convergent*, conversely if $(\sum |z_n|)$ diverges, the series is *conditionally convergent.*

We must also introduce the concept of a *divergent direction.* A divergent direction of a series $\sum(z_n)$ is an angle θ where the points of the sequence elements bunch together. The points that have an argument close to the angle form a divergent subsequence of $\sum(z_n)$. We state this result more formally.

Definition 2. A **divergent direction** of a conditionally convergent series $\sum(z_n)$ is an angle θ such that for any $\epsilon > 0$,

$$\sum_{|\mathrm{Arg}(z_n) - \theta \mod 2\pi| < \epsilon} z_n$$

is properly divergent. The set of (z_n) that are not part of a divergent direction either are finite, or form an absolutely convergent series.

Divergent directions enable us to characterize our series, and to determine which of the three cases a series falls into.

Lemma 3. *If $\sum(z_n)$ is a conditionally convergent series of complex numbers, it has at least two divergent directions.*

Proof. Assume to the contrary that $\sum(z_n)$ has only a single divergent direction, θ. Then, taking the terms of $\sum(z_n)$ that are not a part of a divergent direction, we have either a finite number of terms, or an absolutely convergent series, and therefore their sum equals some finite number a. Since the remaining terms are properly divergent, the series as a whole is properly divergent. □

Example 4. The sequence $(z_n) := (\frac{1}{2^n} + \frac{(-1)^{n+1}}{n} i)$ has two divergent directions: $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

Example 5. The sequence $(z_n) := \frac{e^{i\pi \frac{n}{2}}}{n}$ has four divergent directions: $\frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 0.

In Example 5, every sequence element was exactly on a divergent direction. This is not necessary, as shown by Example 4, where no elements had exactly the same argument as any divergent direction.

3 Results

First, we will show that if $\sum(z_n) \to 0$ is conditionally convergent we can remove a finite number of elements from (z_n) to get a new sequence which can be rearranged to still converge to zero.

Theorem 6 (Rearrangement of series that converge to 0). *Let $\sum(z_n)$ be a conditionally convergent series of complex numbers that converges to 0 and let E be a finite set of elements of (z_n). Then there exists a rearrangement $\sigma(n)$ such that $\sum(z_{\sigma(n)}) \setminus E \to 0$.*

Proof. It suffices to prove the result for if a single element is removed, as the general case follows by induction. Suppose that $\sum(z_n)$ converges conditionally to 0. Then, for every $\epsilon > 0$, there exists a $M \in \mathbb{N}$ such that if $m \geq M$ then, $|\sum_{n=1}^{m} z_n| < \epsilon$ and for every $j \geq m, |\sum_{n=m}^{j} z_n| < \epsilon$. The key to the proof is that given any nonzero complex number $y \in (z_n)$, we can find a subset of elements $\{z_{n_j}\}_y$ that will decrease the modulus of y when added to it, i.e. for any $z_j \in \{z_{n_j}\}_y$, $|z_j + y| < |y|$.

Let z_k denote the element of (z_n) we are removing. Define $\sigma(1) := 1$. Then let z_l be the largest element of $\{z_{n_j}\}_{z_k}$ and define $\sigma(2) := l$. Continue in this way, defining $z_\sigma(n) := \max(z_{n_j})_{\sum_{b=1}^{n-1} z_{\sigma(b)}}$. We have that $(|\sum z_{\sigma(n)}|)$ is a decreasing sequence of real numbers bounded below by 0, so $(|\sum z_{\sigma(n)}|) \to 0$. Thus, we see that if a finite number of elements of (z_n) are removed, there exists a rearrangement of (z_n) such that $(\sum z_{\sigma(n)}) \to 0$. □

This proof is most useful for the following corollary.

Corollary 7. *Removing a finite number of elements from the tail of a conditionally convergent series need not affect its convergence.*

Since the tail of a convergent series must converge to 0, this follows directly from Theorem 6. This means that we can now bring elements from the tail of a sequence to the beginning and still have a convergent sequence; effectively allowing us to add elements from the tail to the limit of the sequence. Now using this, we can begin prove our main result.

Lemma 8. *If $(\sum z_n) \to 0$ and Arg(c) is a divergent direction of (z_n), there exists a rearrangement of (z_n) that converges to c.*

Proof. Given $\epsilon > 0$, let $\sum z_n$ be a conditionally convergent series such that $(\sum z_n) \to 0$, and let $c \in \mathbb{C}$ such that $\text{Arg}(c) = \theta$, where θ is a divergent direction of $\sum z_n$. Then, let M be such that if $m \geq M$, $|\sum_{n=1}^{m} z_n| < \frac{\epsilon}{4}$. Let $\{p_1, p_2, p_3, ..., p_{k-1}, p_k : |\arg(p_n) - \theta| < \epsilon\}$ be elements of the m-tail of z_n such that $|\sum_{i=1}^{k-1} p_i| < |c| < |\sum_{i=1}^{k} p_i|$. Then, define $z_{\pi(n)}$ to be z_n for $n \leq m$, $z_{\pi(n+x)} := p_x$ for $1 \leq x \leq k$, and $z_{\pi(j)}$ to be a rearrangement of the $m-$tail where all partial sums are less than $\epsilon/4$. This rearrangement exists by Corollary 7. Then, $(\sum z_{\pi(n)}) \to c$. □

Corollary 9. *If* $(\sum z_n) \to a$ *and Arg(c−a) is a divergent direction of* z_n*, then there exists a rearrangement of* z_n *that converges to c.*

Example 10. Because $\sum(z_n) := \sum_{j=1}^{n}(\frac{1}{2^j} + \frac{(-1)^{j+1}}{j}i)$ converges to $1+\ln(2)i$ and $\frac{\pi}{2}$ is a divergent direction, there exists a rearrangement of (z_n) that converges to $1+i$ since $\text{Arg}(1+\ln(2)i-(1+i))$=$\text{Arg}((\ln(2)-1)i)$=$\frac{\pi}{2}$.

Lemma 11. *If* $(\sum z_n) \to a$ *conditionally and has exactly 2 divergent directions, then those divergent directions are* θ *and* $-\theta$

We expect this to be true simply because if the two directions didn't cancel one another out, the sequence would necessarily diverge. That is, if one divergent direction were 0 and the other were not π, the imaginary part would be unbounded and thus could not converge. However, as with many other things in this thesis, this proof is much easier said than done.

Proof. Let $(\sum z_n) \to a$ conditionally and have exactly two divergent directions: θ_1 and θ_2. Assume towards a contradiction that $\theta_1 \neq \pm\theta_2$. We may assume without loss of generality that $\theta_1 = 0$ and $\theta_2 < \pi$. Since $(\sum z_n) \to a$, there exists a K such that if $M \geq K$ then $|\sum_{K}^{M} z_n| < 1$. Let (z_{n_k}) denote the subsequence of (z_n) along the divergent direction θ_2, and let $N \geq K$ be such that the imaginary portion of $\sum_K^N z_{n_k} > i$. Then we have $|\sum_{K}^{N} z_n| = |c+i| > 1$, a contradiction. Therefore, we must have that $\theta_2 = -\theta_1$, a fact we now use. □

Lemma 12. *If* $(\sum z_n) \to a$ *and* $(\sum z_{\sigma(n)}) \to b$ *and Arg(a−b) is not a divergent direction of* $\sum(z_n)$*, then* $\sum(z_n)$ *has at least 3 divergent directions.*

Proof. Suppose to the contrary that z_n has only two divergent directions, $\{\theta, -\theta\}$. Then, since $(\sum z_n) \to a$, for every $\epsilon > 0$, there exists some M such that when $m \geq M$, $|\sum_{n=1}^{m} z_n - a| < \epsilon$. Since there are only a finite number of elements who are not part of a divergent direction, let m be great enough to include all of those elements. Furthermore, there exists an l such that $\left|\sum_{n=1}^{l} z_{\sigma(n)} - b\right| < \epsilon$ and $\{z_{\sigma(n)}\}_{n=1}^{l} \supset \{z_n\}_{n=1}^{m}$. Now, we look at the set X of elements that are in $\{z_{\sigma(n)}\}_{n=1}^{l}$ but not in $\{z_n\}_{n=1}^{m}$. Since $|\sum_{n=1}^{m}\{z_n\} - a| < \epsilon$ and $\left|\sum_{n=1}^{l} z_{\sigma(n)} - b\right| < \epsilon$, $\sum X = (b-a)$. However, this would require X to contain at least one element x where $\text{Arg}(x) \neq \pm\theta$, contradicting our hypothesis that $\{z_n\}_{n=1}^{m}$ contained all such elements. Therefore, (z_n) has more than 2 divergent directions. □

Finally, we are ready for our main result.

Theorem 13. *If* $(\sum z_n) \to 0$ *and* (z_n) *has at least three divergent directions then for all* $c \in \mathbb{C}$ *there exists a rearrangement* $z_{\pi(n)}$ *such that* $(\sum z_{\sigma(n)}) \to c$*.*

Proof. Since there are at least three distinct divergent directions, they must span $\mathbb{C}$, i.e. $\forall c \in \mathbb{C}$, $c = r_1e^{i\theta_1} + r_2e^{i\theta_2}$ where θ_1 and θ_2 are divergent directions of (z_n). Since $\text{Arg}(r_1e^{i\theta_1})$ is a divergent direction of (z_n), there exists a rearrangement $(z_{\pi(n)})$ such that $\sum(z_{\pi(n)}) \to r_1e^{i\theta_1}$ by Lemma 8. Then, since $\sum(z_{\pi(n)}) \to r_1e^{i\theta_1}$ and $\text{Arg}(c - r_1e^{i\theta_1})$=$\text{Arg}(r_2e^{i\theta_2})$=$\theta_2$ is a divergent direction of $(z_{\eta(n)})$, by Corollary 9, there exists a rearrangement $(z_{\sigma(n)})$ such that $\sum(z_{\sigma(n)}) \to c$. □

Since we have proven our result for $\mathbb{C}$, we have also proven it for $\mathbb{R}^2$. Now, we can focus on proving our result for $\mathbb{R}^n$. We need to first generalize our definition of a divergent direction. Any vector in $\mathbb{R}^n$ can be expressed as a magnitude along with $n-1$ angles: the rotation from each axis. We will refer to all of these angles as a single vector of $n-1$ angles $\boldsymbol{\theta}$ and say that a vector $\vec{x}$ has angle $\boldsymbol{\theta}$ if each of the $n-1$ angles of $\vec{x}$ are equal to the angles specified in $\boldsymbol{\theta}$.

Example 14. In $\mathbb{R}^3$, the vector $(-1, 1, -\sqrt{2})$ in Cartesian coordinates is equal to $(2, \frac{3\pi}{4}, \frac{3\pi}{4})$ in spherical coordinates. Thus, its angle $\boldsymbol{\theta}$ is equal to $(\frac{3\pi}{4}, \frac{3\pi}{4})$.

We denote the unit vector with direction $\boldsymbol{\theta}$ by $\hat{u}_\theta$. Our concept of divergent direction remains the same: a divergent direction of a sequence (z_k) in $\mathbb{R}^n$ is an angle $\boldsymbol{\theta}$ such that for any $\epsilon > 0$, $\displaystyle\sum_{|\arg(z_k - \boldsymbol{\theta}|<\epsilon} z_k$ is properly divergent.

Example 15. The sequence in $\mathbb{R}^3$ defined by $(\frac{1}{2^n}, \frac{(-1)^n}{n}, \frac{(-1)^{n+1}}{n})$ has divergent directions $(\frac{\pi}{4}, \frac{\pi}{2})$ and $(\frac{\pi}{4}, \frac{3\pi}{2})$.

Our key theorem, Theorem 6, did not rely on any properties of $\mathbb{C}$ or $\mathbb{R}^2$ that are not also found in $\mathbb{R}^n$, and so is instantly generalizable to $\mathbb{R}^n$. In fact, Theorem 8 also generalizes to $\mathbb{R}^n$. Thus, we need to only change our conclusion.

Theorem 16. *If $\sum(z_k)$ converges conditionally to $\mathbf{0}$ and has divergent directions $\boldsymbol{\theta_1}, \boldsymbol{\theta_2}, \cdots, \boldsymbol{\theta_n}$, then for any $c \in$ span$(\hat{u}_{\boldsymbol{\theta_1}}, \cdots, \hat{u}_{\boldsymbol{\theta_n}})$ there exists a rearrangement $(z_{\sigma(k)})$ such that $(\sum z_{\sigma(k)}) \to c$.*

Proof. Since $c \in \text{span}(\hat{u}_{\boldsymbol{\theta_1}}, \cdots, \hat{u}_{\boldsymbol{\theta_n}})$, we know $c = a_1\hat{u}_{\boldsymbol{\theta_1}} + \cdots + a_n\hat{u}_{\boldsymbol{\theta_n}}$. By Theorem 7 we know that there exists a rearrangement $(z_{\eta(k)})$ such that $\sum(z_{\eta(k)}) \to a_1\hat{u}_{\boldsymbol{\theta_1}}$. This implies that there exists an M such that if $m \geq M$ we have that $|\displaystyle\sum_{k=1}^{m} z_{\eta(k)} - a_1\hat{u}_{\boldsymbol{\theta_1}}| < \frac{\epsilon}{n}$ and $|\displaystyle\sum_{k=m}^{\infty} z_{\eta(k)}| < \frac{\epsilon}{n}$. Since the m-tail of $(z_{\eta(k)})$ converges conditionally to 0 and $\boldsymbol{\theta_2}$ is a divergent direction, we know by Theorem 7 that there exists a rearrangement $(z_{\xi(k)})$ such that $\sum z_{\xi(k)} \to a_2\hat{u}_{\boldsymbol{\theta_2}}$. Considering the sequence as a whole again, we find that with our new rearrangement $z_{\pi(n)}$ we have $\sum z_{\pi(n)} \to a_1\hat{u}_{\boldsymbol{\theta_1}} + a_2\hat{u}_{\boldsymbol{\theta_2}}$. Thus, by induction, we have that there exists a rearrangement $(z_{\sigma(k)})$ such that $\sum(z_{\sigma(k)}) \to c$. □

4 Conclusion and Directions for Further Research

We have found the conditions required to use Riemann's Rearrangement Theorem in higher dimensions. Unlike in $\mathbb{R}$, being conditionally convergent in $\mathbb{C}$ is not enough to ensure that it is possible to rearrange the series to converge to any element of $\mathbb{C}$. We instead require that our sequence have enough divergent directions to ensure it can be rearranged to the desired complex number. The only three possibilities for any series are that all rearrangements converge to the same value, all rearrangements converge to elements on a line, or all complex numbers are possible to rearrange to. In $\mathbb{R}^n$, our results still hold in general, but there are more possibilities beyond a single value, lines, and any points: in fact, for any subspace of $\mathbb{R}^n$, there is a series that can be arranged to converge to any value in that subspace and no other values.

For further research, infinite dimensional spaces such as $\mathbb{R}^{\mathbb{N}}$ or function spaces could be examined. Determining the effect of rearranging a conditionally convergent series of functions could introduce several new questions depending on whether the convergence is uniform or pointwise, and dependence on the metric used. The existence of similar rearrangement theorems for function spaces would also have interesting implications in power series of functions. However, do to the added complexity of working with functions that have an uncountable domain, it is unlikely that such theorems exist or that they can be proven similarly to the rearrangement theorems for finite dimensional spaces.