Euler's Smallest Squares

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Abstract

Leonhard Euler approached a problem involving x, y, and z for which $x^2 + y^2 + z^2$ and $x^2y^2 + y^2y^2 + x^2z^2$ are perfect squares. Euler believed that he had found the absolute smallest solution to this problem. However, Euler made mathematical mistakes when he was calculating the formulas he used to derive the trios. Thus, depending on the specified definition of the word small, smaller solutions that satisfy the original problem were discovered. We will find that some definitions of small do in fact have smaller solutions then the ones that Euler found, and that Euler's methods when approaching this problem may not be the most efficient.

1 Introduction

Imagine being a famous and accomplished mathematician who went blind after an operation to remove cataracts. This is the situation that Leonhard Euler found himself in late in his life. Euler was a mathematician in the 18th century known for his discoveries in geometry, trigonometry, number theory and calculus. Euler approached a problem in Volume VIII of the *Novi Commentarii* that examines x, y, and z whose sums calculated using different definitions of small are perfect squares. Euler used four different variations of his original technique to find five solutions, one of which he deemed to be the absolute smallest. However, perhaps due to his blindness, Euler made multiple computational mistakes in his calculations. We will be using Euler's four formulas and modern technology to determine whether or not Euler did in fact find the smallest possible solution and if there are any other solutions or methods that can be discovered using Euler's formulas.

2 Definitions and Development

In Article 523 in the Euler Archive [1], Leonhard Euler examined sets of three numbers whose sum was a perfect square in order to discover the absolute smallest solution. These variables had to satisfy two conditions,

$$xx + yy + zz = \text{perfect square}$$

and

$$xxyy + xxzz + yyzz = \text{perfect square.}$$

Euler solved these equations by setting $x = pp + qq - rr$; $y = 2pr$, and $z = 2qr$, so that

$$xx + yy + zz = (pp + qq + rr)^2.$$

Now, consider the second condition and suppose that $xxyy + xxzz + yyzz = Q^2$. Euler substituted in for x, y, and z in order to get the equation

$$4rr(pp + qq)(pp + qq - rr)^2 + 16ppqqr^4 = Q^2,$$

such that, when we divide by a square factor of $4rr$, we will get

$$(pp + qq)(pp + qq - rr)^2 + 4ppqqrr = \frac{QQ}{4rr}.$$

However, the left side of the previous equation must be rendered as a square, so Euler introduced another substitution with a new variable $r = p - nq$. After substituting in this assumption and simplifying the equation, Euler states that the ending equation is sufficient to satisfy the second condition. Now the variables p and q must be written in terms of n. To accomplish this, Euler created a ratio between p and q such that

$$\frac{p}{q} = \frac{8n(1-nn)}{5-10nn+n^4}.$$

Details of this computation can be found in [1]. From the ratio, Euler was able to set $q = 5 - 10nn + n^4$ and $p = 8n(1 - nn)$. Now, take the assumption $r = p - nq$ and substitute in for p and q. From this we learn that $r = n(3 + 2nn - n^4)$. By setting $n = 3$, Euler was able to generate the solution $x = 28; y = 432; z = 9$ which he states is the trio that generates the smallest possible solution.

Euler then looked at three other test cases that led him to the equations

$$p = 2n(4 + 2m - 2mm - m^3), q = 4 + 8m - 5mm - 4m^3, r = n(4 - 4m + mm + 2m^3);$$

$$p = m^8 - 2m^4n^2(n^2 - 3) + n^4(n^2 - 3)^2 - 16m^2(n^2 - 1)^2, q = 8m^2n(8 + m^4 - 5n^2 - n^4),$$

$$r = m^2(m^6 + 16) + n^4(n^2 - 3)^2 - 2m^2n^2(m^2n^2 + 3m^2 + 48 + 4m^6) + 8m^2n^4(3 + n^2);$$

and

$$p = -(4 - 2m - mm); q = 4n; r = -4 + 2m + mm - 4n^2.$$

Using the last set of equations, Euler actually miscalculated one of his solutions. This is documented in the translation of Article 523 by Snavely and Woodruff and is explained a bit later in this thesis. With these variables we can calculate a number of solutions. While calculating solutions, Euler stated that the third set of equations listed above did not reveal trios that generate solutions smaller than the solutions he had already found. Because of this, Euler did not focus on the solutions from this set of equations and neither will we. Euler then went through four more examples in which he found solutions that satisfy the two conditions. The trios for these solutions consist of

$$x = 513, y = 19, z = 76;$$

$$x = 2583, y = 2112, z = 1804;$$

$$x = 196, y = 693, z = 528;$$

and

$$x = 468, y = 37, z = 1776.$$

However, Snavely and Woodruff found another mistake in the calculation of the trio $x = 468$, $y = 37, z = 1776$, noting that the second condition is not satisfied. The corrected numbers are $x = 108, y = 7, z = 336$. This mistake is documented in the translation of Article 523. Euler then went one step further and stated that the solution generated from the trio $x = 28, y = 432, z = 9$ is “without a doubt the smallest” solution that satisfies the formulas. In this thesis, we will determine if Euler did in fact find the absolute smallest solution and if not, would he have eventually found it.

3 Results

It is first important to mention that Euler never stated what definition of small he was using when looking at solutions. Therefore, we analyzed solutions based in seven definitions of small as shown in Table 1.

Min (x, y, z)	Min (xyz)
Max (x, y, z)	Min $\left(\frac{1}{3}(x + y + z)\right)$
Min $(x^2 + y^2 + z^2)$	Max (x, y, z) - Min (x, y, z)
Min $(x^2y^2 + x^2z^2 + y^2z^2)$	

Table 1: Definitions of Small

After calculating solutions from Euler’s formulas based on our definitions of small we discovered that there were multiple solutions that Euler did not document in his article. Due to this, we decided to generate a list of all possible solutions for $x, y, z \leq 1000$. Using the C++ programing language, we created a program that uses brute force to generate trios that satisfy Euler’s two conditions. This program can be seen in the appendices.

The program was able to generate 40 trios that meet Euler’s two conditions when $1 \leq x \leq y \leq z \leq 1000$. These trios can be seen in the appendices. Of those 40 trios, excluding the five trios that Euler already documented, there are five trios that generate solutions that could be considered the smallest or close to the smallest solution depending on the definition of small. The trios are

$$x = 14, y = 216, z = 672;$$

$$x = 18, y = 56, z = 864;$$

$$x = 35, y = 72, z = 96;$$

$$x = 36, y = 427, z = 672;$$

and

$$x = 44, y = 57, z = 144.$$

Of the trios found using Euler's formulas, or what we will refer to as the Euler trios, and those not found using his formulas, Tables 2-8 display the results of the smallest solutions for our seven definitions of small. As can be seen in the tables below, Euler's smallest trio $x = 28; y = 432; z = 9$ is not always the smallest.

Placement	Euler Solution	Euler Trio	Best Solution	Best Trio
1	7	(7, 108, 336)	7	(7, 108, 336)
2	9	(9, 28, 432)	9	(9, 28, 432)
3	36	(36, 427, 672)	14	(14, 216, 672)

Table 2: Min (x, y, z)

Placement	Euler Solution	Euler Trio	Best Solution	Best Trio
1	144	(44, 57, 144)	96	(35, 72, 96)
2	336	(7, 108, 336)	144	(44, 57, 144)
3	432	(9, 28, 432)	336	(7, 108, 336)

Table 3: Max (x, y, z)

Placement	Euler Solution	Euler Trio	Best Solution	Best Trio
1	25,921	(44, 57, 144)	15,625	(35, 72, 96)
2	124,609	(7, 108, 336)	25,921	(44, 57, 144)
3	187,489	(9, 28, 432)	124,609	(7, 108, 336)

Table 4: Min $(x^2 + y^2 + z^2)$

Placement	Euler Solution	Euler Trio	Best Solution	Best Trio
1	113,806,224	(44, 57, 144)	65,415,744	(35, 72, 96)
2	161,493,264	(9, 28, 432)	113,806,224	(44, 57, 144)
3	1,322,922,384	(7, 108, 336)	161,493,264	(9, 28, 432)

Table 5: Min $(x^2y^2 + x^2z^2 + y^2z^2)$

Placement	Euler Solution	Euler Trio	Best Solution	Best Trio
1	245/3	(44, 57, 144)	203/3	(35, 72, 96)
2	451/3	(7, 108, 336)	245/3	(44, 57, 144)
3	469/3	(9, 28, 432)	451/3	(7, 108, 336)

Table 6: Min $\left(\frac{1}{3}(x+y+z)\right)$

Placement	Euler Solution	Euler Trio	Best Solution	Best Trio
1	108,864	(9, 28, 432)	108,864	(9, 28, 432)
2	254,016	(7, 108, 336)	241,920	(35, 72, 96)
3	361, 152	(44, 57, 144)	254,016	(7, 108, 336)

Table 7: Min (xyz)

Placement	Euler Solution	Euler Trio	Best Solution	Best Trio
1	100	(44, 57, 144)	61	(35, 72, 96)
2	329	(7, 108, 336)	100	(44, 57, 144)
3	368	(64, 171, 432)	329	(7, 108, 336)

Table 8: Max (x, y, z) - Min (x, y, z)

Thus, as shown in Tables 2-8, Euler's smallest solution is in fact only considered the smallest for two of our definitions of small. Even Euler's corrected trio $x = 108, y = 7, z = 336$ is only considered to generate the smallest possible solution for one of our definitions of small. Also shown in Tables 2-8, there are indeed trios that satisfy the two conditions, but have not as of yet been produced using Euler's formulas. The most notable trio is $x = 35, y = 72, z = 96$ which is actually the smallest solution for all of our definitions of small except for Min (x, y, z) and Min (xyz). This leads us to the conclusion that Euler's methods used to address this problem may not be the most efficient methods possible.

4 Conclusion and Directions for Further Research

We must now address our certainty of finding the smallest possible solution for each definition of small. For the definition Min (x, y, z) we cannot state that we found the smallest possible solution with complete certainty because there is a possibility of a smaller solution past the 1000 mark that is used in our computer program; however we are fairly confident that no such smaller solution exists. It is also good to note that we believe the definition Min (x, y, z) follows the pattern in which as the values of x, y or z grow, so do the other values. However, we cannot say with absolute certainty that we have found the without a doubt smallest solution for this definition of small. We do however have complete certainty that we have found the smallest solution for the definition Max (x, y, z). As we have already explained, through our computer program we have found the trios that hold the smallest possible values for x, y and z. Thus, we have found the trio that generates the smallest possible solution for Max (x, y, z) because there will not be any smaller values for x, y and z due to the pattern followed in this definition.

The definitions Min $(x^2 + y^2 + z^2)$ and Min $\left(\frac{1}{3}(x + y + z)\right)$ also have the smallest possible solutions with complete certainty. For definition Min $(x^2 + y^2 + z^2)$ we will consider the trio (0, 0, 1001) which falls outside of the bounds in the computer program. The measure of smallness generated by this trio using the definition Min $(x^2 + y^2 + z^2)$ is 1,002,001; which is larger than any solutions found in the computer program which has a limit of 1000. Thus, we have found the smallest possible solution for the definition Min $(x^2 + y^2 + z^2)$. The same trio (0, 0, 1001) can be used for the definition Min $\left(\frac{1}{3}(x + y + z)\right)$. The solution generated by this trio using this definition is larger than any other solutions found and will also continue to increase as the values of x, y and z increase. Therefore, we have found the absolute smallest solution for the definition Min $\left(\frac{1}{3}(x + y + z)\right)$.

For the definitions Min $(x^2y^2 + x^2z^2 + y^2z^2)$ and Min (xyz) we have varying degrees of certainty on the solutions we have found. We believe, based on the pattern that the solutions to the definitions in this paper follow, that we have found the absolute smallest solution for Min $(x^2y^2 + x^2z^2 + y^2z^2)$ and Min (xyz). However, we have yet to find a definite proof to solidify our beliefs on the solutions with these definitions of small. We do not hold any certainty with the solutions in the definition Max (x, y, z) - Min (x, y, z). It is possible that there is a solution generated by a trio that has exceptionally large values for x, y and z and yet have a smaller distance then a trio that has the smallest possible values for x, y and z. Therefore, we have varying degrees of certainty on finding the smallest possible solutions depending on the definition of small.

As already revealed, Euler did not find the “without a doubt” [1] smallest solution to his original problem. He could have found some of the solutions that were in fact smaller than his with the approach he took; however, there were multiple solutions generated in the computer program that are in fact smaller. Therefore it is possible that Euler’s approach was not the most efficient approach because it misses significant solutions. For future work we would like to analyze Euler’s formulas with additional definitions of small, have complete certainty of our results for each definition of small, prove without a doubt that Euler’s approach is not efficient, and attempt to discover a possible better approach to the original problem.

References

[1] Snavely, Mark R., Woodruff, Phil. *On three square numbers, of which the sum and the sum of products two apiece will be a square (De Tribus Numeris Quadratis, quorum tam summa, quam summa productorum ex binis sit quadratum)* Translation from Latin into English. Available online as E523 at www.eulerarchive.org.

Appendices

Appendix 1: Computer Program for Generating Trios

```
#include <iostream>
#include <cmath>
```

```
#include <vector>

using namespace std;

int main()
{
	vector<int> solutions1;
	vector<int> solutions2;
	vector<int> positionX;
	vector<int> positionY;
	vector<int> positionZ;

	for (double i = 1.0; i <= 1000.0; i++)
	{
		for (double j = i; j <= 1000.0; j++)
		{
			for (double k = j; k <= 1000.0; k++)
			{
				if (sqrt((i*i) + (j*j) + (k*k)) == floor(sqrt((i*i) + (j*j) + (k*k)))) //whole number
				{
					if (sqrt(((i*i)*(j*j)) + ((i*i)*(k*k)) + ((j*j)*(k*k))) ==
floor(sqrt(((i*i)*(j*j)) + ((i*i)*(k*k)) + ((j*j)*(k*k))))) //whole number
					{
						solutions1.push_back(sqrt((i*i) + (j*j) + (k*k)));
						solutions2.push_back(sqrt(((i*i)*(j*j)) + ((i*i)*(k*k)) +
((j*j)*(k*k))));

						positionX.push_back(i);
						positionY.push_back(j);
						positionZ.push_back(k);

					}
				}
			}
		}
	}

	for (int p = 0; p < solutions1.size(); p++)
	{
		cout << "The solutions " << solutions1[p] << " and " << solutions2[p] << " are from (" <<
positionX[p] << " , " << positionY[p] << " , " << positionZ[p] << ")." << endl;
	}

	system("PAUSE");
}
```

Appendix 2: Computer Generated Trios

(7, 108, 336)	(9, 28, 432)	(14, 216, 672)	(18, 56, 864)
(35, 72, 96)	(36, 427, 672)	(44, 57, 144)	(64, 171, 432)
(70, 144, 192)	(76, 192, 513)	(87, 144, 364)	(88, 114, 288)
(105, 140, 288)	(105, 216, 288)	(112, 147, 396)	(128, 342, 864)
(132, 171, 432)	(140, 288, 384)	(174, 288, 728)	(175, 360, 480)
(176, 228, 576)	(192, 252, 301)	(196, 528, 693)	(209, 528, 684)
(210, 280, 576)	(210, 432, 576)	(220, 285, 720)	(224, 294, 792)
(245, 504, 672)	(264, 342, 864)	(280, 576, 768)	(288, 316, 357)
(288, 604, 627)	(315, 420, 864)	(315, 648, 864)	(350, 720, 960)
(384, 504, 602)	(576, 632, 714)	(576, 688, 903)	(576, 756, 903)